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# Détermination sous-différentielle, propriété Radon-Nikodým de faces, et structure différentielle des ensembles prox-réguliers

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*To my father,*
*the person I've always admired the most.*

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# Contents

# Introduction Générale

Cette thèse est consacrée à l'étude de la géométrie de divers types d'ensembles dans le contexte d'espaces de Banach, ainsi qu'à des propriétés analytiques de certaines classes de fonctions associées à de tels ensembles et qui sont définies sur les espaces de Banach correspondants. Les propriétés géométriques de ces ensembles varient suivant les espaces de Banach considérés, de sorte qu'en général les résultats fondamentaux nécessitent des hypothèses appropriées sur l'espace de Banach lui-même. Pour cette raison, ce travail est divisé en deux parties: La première est relative à certaines études géométriques des ensembles convexes dans des espaces de Banach (Géométrie convexe), et la seconde à de nouvelles caractéristiques, dans les espaces de Hilbert, de la géométrie des ensembles prox-réguliers dont la projection métrique est différentiable sur des couronnes ouvertes spécifiques (Géométrie non convexe). Dans cette introduction générale nous présentons quelques unes de nos contributions dans les deux sujets ci-dessus, qui sont contenu en [10], [12], [29], [30].

## Partie I: Géométrie Convexe

La première partie de la thèse considère le problème de détermination de fonctions à partir du *sous-différentiel de Moreau-Rockafellar*. Rappelons que, étant donné un espace localement convexe $(X, \theta)$, une fonction $f : X \to \overline{\mathbb{R}}$ (où $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$) et un point $x \in X$, le sous-différentiel de $f$ en $x$ est donné par

$$\partial f(x) = \begin{cases} \{x^* \in X^* \ : \ f(x) + \langle x^*, y - x\rangle \leq f(y), \ \forall y \in X\} & \text{si } f(x) \in \mathbb{R} \\ \emptyset & \text{autrement,} \end{cases} \tag{1}$$

où $X^*$ désigne le dual topologique de $(X, \theta)$. Dans les années 60, J.J. Moreau [23] (dans le contexte des espaces de Hilbert) et ensuite R. T. Rockafellar [28] (dans le cas d'espaces de Banach) ont démontré le résultat célèbre suivant d'intégration des fonction convexes:

**Théorème 1** (Moreau-Rockafellar) *Soit $X$ un espace de Banach et soit un couple de fonctions $f, g : X \to \mathbb{R}_\infty$ (où $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$) qui sont convexes, propres et semi-*

*continues inférieurement. Si l'on a*

$$\partial f(x) \subseteq \partial g(x), \quad \forall x \in X,$$

*alors les fonctions $f$ et $g$ coïncident à une constante additive près, c'est-à-dire, il existe une constante $c \in \mathbb{R}$ telle que $f = g + c$.*

Suite à cette contribution, à partir de la seconde moitié des années 80 plusieurs études ont été réalisées en vue de généraliser ce théorème, soit en relaxant l'hypothèse de convexité, soit en remplaçant $X$ par un espace localement convexe général, soit en utilisant d'autres notions de sous-différentiels en présence de non-convexité. On renvoie aux articles [1], [3], [11], [21], [33]–[35] pour plus d'information à propos du développement de la théorie de détermination/intégration d'une fonction à partir de son sous-différentiel.

Une de ces contributions est constituée de l'article de R. Correa, Y. García et A. Hantoute [9], où les auteurs établissent diverses formules d'intégration pour une large sous-famille de fonctions (pas nécessairement convexes), qui a été introduite par J. Benoist et J.-B. Hiriart-Urruty dans [18]: Les fonctions *épi-pointées*. Pour un espace de Banach $X$, une fonction $f : X \to \mathbb{R}_\infty$ est dite être épi-pointée quand

$$\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset,$$

où $f^*$ désigne la conjuguée (de Legendre-Fenchel) de $f$. Le concept de fonction épi-pointée a été au début définie à partir d'une notion de "coercivité". La définition précédente est sa caractérisation la plus connue et probablement la plus utilisée.

Le résultat principal de [9] peut être énoncé comme suit:

**Théorème 2** (Correa-García-Hantoute, 2012) *Soit $X$ un espace de Banach qui a la propriété de Radon-Nikodým, et soit $f : X \to \mathbb{R}_\infty$ une fonction épi-pointée et semi-continue inférieurement. Pour toute autre fonction $g : X \to \overline{\mathbb{R}}$ telle que*

$$\partial f(x) \subseteq \partial g(x), \quad \forall x \in X,$$

*il existe une constante $c \in \mathbb{R}$ telle que*

$$\overline{\operatorname{co}}\, f = \overline{(\overline{\operatorname{co}}\, g) \square \sigma_{\operatorname{dom} f^*}} + c,$$

*où $\overline{\operatorname{co}}\, f$ désigne l'enveloppe convexe fermée de $f$, $\square$ désigne l'opération d'inf-convolution de Moreau, et $\sigma_{\operatorname{dom} f^*}$ est la fonction d'appuie de* $\operatorname{dom} f^*$.

Rappelons qu'un espace de Banach $X$ a la *propriété de Radon-Nikodým* (RNP) si pour tout sous-ensemble $K$ convexe, fermé et borné de $X$, l'égalité

$$K = \overline{\operatorname{co}}(\text{str-exp}(K)), \tag{2}$$

a lieu, où str-exp$(K)$ désigne l'ensemble des points *fortement-exposés* de $K$. Un point $x \in K$ est dit être fortement-exposé s'il existe une forme linéaire continue $x^* \in X^*$ telle que

(a) $\langle x^*, x\rangle = \sigma_K(x^*) > \langle x^*, y\rangle$, pour tout $y \in K \setminus \{x\}$; et

(b) Pour toute suite $(x_n)$ de $K$,

$$\langle x^*, x_n\rangle \to \sigma_K(x^*) \implies \|x - x_n\| \to 0.$$

Historiquement, la propriété RNP a été introduite comme une propriété reliée à des mesures à valeurs vectorielles et à l'existence de leurs dérivées, au sens du théorème classique de Radon-Nikodým. Cependant, plusieurs caractérisations de cette propriété ont été établies (dont celle ci-dessus), donnant des interprétations géométriques très claires qui ont permis le développement de cette théorie au niveau de l'Analyse Fonctionnelle et de la Géométrie des Espaces de Banach. Pour plus de détails, le lecteur peut consulter le livre de R. Bourgin [2], ou celui de J. Diestel et J.-J. Uhl [13].

L'observation principale de Correa, García et Hantoute dans leur article [9] qui a conduit au théorème 2 ci-dessus est la suivante: Quand un espace de Banach $X$ a la propriété RNP, alors pour toute fonction $f : X \to \mathbb{R}_\infty$ qui est épi-pointée et semi-continue inférieurement, il existe un ensemble $D$ dense dans $\operatorname{int}(\operatorname{dom} f^*)$ tel que

$$\partial f^*(x^*) = (\partial f)^{-1}(x^*) \quad \text{pour tout } x^* \in D. \tag{3}$$

Malheureusement, l'équation (3) devient trop forte quand les espaces ne jouissent pas de la propriété RNP. Le premier objectif du premier chapitre de cette thèse consiste à relaxer (3) et à obtenir une formule d'intégration analogue à celle du Théorème 2 dans le contexte des espaces localement convexes.

## Chapitre 1: Formule d'Intégration

Rappelons que, étant donné un espace (de Hausdorff) localement convexe $(X, \theta)$ et son dual topologique $X^*$, il existe plusieurs topologies (de Hausdorff) localement convexes reliées à la dualité $\langle X, X^*\rangle$. Dans ce que suit, on désignera par $w(X, X^*)$ (ou simplement, par $w$) la topologie faible sur $X$ induite par $X^*$, et par $w^*(X^*, X)$ (ou simplement, par $w^*$) la topologie faible-étoile sur $X^*$ induite par $X$.

On désignera également par $\tau(X, X^*)$ la topologie de Mackey sur $X$ induite par $X^*$. Rappelons que $\tau(X, X^*)$ est la topologie localement convexe la plus fine sur $X$ qui préserve la dualité $\langle X, X^*\rangle$, c'est-a-dire, qui satisfait $(X, \tau(X, X^*))^* = X^*$. Elle est définie par la convergence uniforme sur les ensembles $w^*$-compacts de $X^*$: Une suite généralisé $(x_i)_{i\in I}$ de $X$ $\tau(X, X^*)$-converge vers un point $x \in X$ si et seulement si pour tout ensemble $K \subseteq X^*$ $w^*$-compact, on a

$$\sup_{x^*\in K} |\langle x^*, x - x_i\rangle| \to 0.$$

De façon similaire, on notera $\tau(X^*, X)$ la topologie de Mackey sur $X^*$ induite par $X$. C'est la topologie localement convexe la plus fine sur $X^*$ qui préserve la dualité $\langle X, X^*\rangle$. Elle est définie par la convergence uniforme sur les ensembles $w$-compacts de $X$. Finalement, on désignera par $\beta(X^*, X)$ la topologie forte sur $X^*$ induite par $X$, c'est-à-dire la topologie sur $X^*$ de la convergence uniforme sur les ensembles bornés de $X$. Rappelons que les ensembles bornés pour n'importe quelle topologie localement convexe entre $w(X, X^*)$ et $\tau(X, X^*)$ sont les mêmes, et donc il n'y pas d'ambiguïté en ne spécifiant pas la topologie quand on parle d'ensembles bornés.

En général, la topologie $\beta(X^*, X)$ est plus fine que $\tau(X^*, X)$, donc elle ne préserve pas nécessairement la dualité $\langle X, X^* \rangle$. On désignera par $X^{**}$, l'espace bidual topologique de $X$, à savoir, $X^{**} := (X^*, \beta(X^*, X))^*$. De la même façon, on peut définir sur $X^*$ les topologies faible $w(X^*, X^{**})$ et de Mackey $\tau(X^*, X^{**})$ induites sur $X^*$ par $X^{**}$. Pour plus d'information sur ces structures topologiques, on renvoie le lecteur au livre de H. Schaefer et M. Wolff [31]. Le diagramme suivant résume les relations entre les différentes topologies ci-dessus sur $X^*$:

$$\underbrace{w^*(X^*, X) \subseteq \tau(X^*, X)}_{\text{dualité } \langle X, X^* \rangle} \quad \text{et} \quad \underbrace{w(X^*, X^{**}) \subseteq \beta(X^*, X) \subseteq \tau(X^*, X^{**})}_{\text{dualité } \langle X^*, X^{**} \rangle}. \tag{4}$$

Désormais, on posera $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, et par $\tau_0$ on désignera la topologie usuelle de $\mathbb{R}$ (et ses extensions naturelles à $\mathbb{R}_\infty$ et $\overline{\mathbb{R}}$). Aussi, suivant la notation originale de Moreau [24], on désignera par $\Gamma_0(X, \theta)$ l'ensemble de toutes les fonctions $f : X \to \mathbb{R}_\infty$ qui sont convexes, propres, et $\theta$-semi-continues inférieurement. On sait que

$$\Gamma_0(X, w) = \Gamma_0(X, \theta) = \Gamma_0(X, \tau(X, X^*)),$$

donc, on écrira simplement $\Gamma_0(X)$ au lieu de $\Gamma_0(X, \theta)$. En même temps, on écrira $\Gamma_0(X^*)$ au lieu de $\Gamma_0(X^*, \beta(X^*, X))$. Finalement, pour une fonction $f : X \to \mathbb{R}_\infty$, on notera $\mathrm{Cont}[f, \theta]$ l'ensemble des points où $f$ est finie et $\theta$-continue. On sait que, pour une fonction $f \in \Gamma_0(X)$, si $\mathrm{Cont}[f, \theta] \neq \emptyset$, alors

$$\mathrm{Cont}[f, \theta] = \mathrm{int}_\theta(\mathrm{dom}\, f).$$

La première contribution du Chapitre 1 est constituée de diverses propriétés du sous-différentiel d'une fonction $f$ de $\Gamma_0(X)$, quand $\mathrm{Cont}[f, \theta]$ n'est pas vide.

Rappelons qu' une multi-application $M : X \rightrightarrows X^*$ est dite *monotone* si

$$\langle x^* - y^*, x - y \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in \mathrm{gph}\, M,$$

où $\mathrm{gph}\, M$ désigne le graphe de $M$ dans $X \times X^*$. Par ailleurs, $M$ est dite *monotone maximale* si pour toute outre multi-application monotone $M'$ telle que $\mathrm{gph}\, M \subseteq \mathrm{gph}\, M'$, on a $M = M'$.

Rappelons aussi que, étant donné $x \in \mathrm{dom}\, M$, $M$ est dite

(a) $\theta$-$w^*$-semi-continue extérieurement en $x$ si pour toute suite généralisé $(x_i, x_i^*)$ dans $\operatorname{gph} M$ telle que $x_i \xrightarrow{\theta} x$ et telle que $x_i^* \xrightarrow{w^*} x^* \in X^*$, on a $x^* \in M(x)$.

(b) $\theta$-$w^*$-semi-continue supérieurement en $x$ si pour tout ensemble $w^*$-ouvert $W$ de $X^*$ contenant $M(x)$, il existe un $\theta$-voisinage $U$ de $x$ telle que $M(U) \subseteq W$.

**Proposition 3** *Soit $(X, \theta)$ un espace localement convexe et soit $f \in \Gamma_0(X)$ telle que* $\operatorname{Cont}[f, \tau(X, X^*)] \neq \emptyset$. *Alors,*

*(a) $\partial f$ est une multi-application monotone maximale.*

*(b) Pour tout $x \in \operatorname{Cont}[f, \tau(X, X^*)]$, $\partial f$ est $\tau(X, X^*)$-$w^*$-semi-continue extérieurement en $x$.*

Dans le contexte des espaces topologiques généraux, les semi-continuités extérieure et supérieure ne sont pas reliées. Par contre, pour des espaces localement convexes, la semi-continuité supérieure implique la semi-continuité extérieure. Aussi, à condition qu'il existe un voisinage de $x$ tel que son image soit précompacte, la semi-continuité extérieure implique la supérieure.

On a redécouvert (dans [10]) les résultats de la Proposition 3. Ces résultats peuvent être trouvés dans [24, Ch. 11-12].

Avec l'objectif de généraliser le théorème 2 au contexte des espaces localement convexes, il nous faut introduire une nouvelle notion de fonctions épi-pointées:

**Définition 4** (Fonction $\tau$-épi-pointée) *Soit $(X, \theta)$ un espace localement convexe et soit $\tau$ une topologie localement convexe sur $X^*$ plus fine que $w^*(X^*, X)$. On dit qu'une fonction $f : X \to \overline{\mathbb{R}}$ est $\tau$-épi-pointée si* $\operatorname{Cont}[f^*, \tau]$ *n'est pas vide.*

Malheureusement, en dehors du contexte des espaces de Banach avec la propriété RNP, la semi-continuité inférieure n'est pas suffisante pour établir une formule d'intégration à partir du sous-différentiel, comme illustré dans la proposition suivante:

**Proposition 5** *Soit $X$ un espace de Banach ne vérifiant pas la propriété RNP. Alors, il existe une fonction $f : X \to \mathbb{R}_\infty$ épi-pointée et semi-continue inférieurement telle que $\partial f(x) = \emptyset$ pour tout point $x \in X$.*

Compte tenu de cette situation, on va remplacer l'hypothèse de semi-continuité inférieure par une nouvelle classe de fonctions qui satisfont un affaiblissement de l'équation (3): Les fonctions *"Subdifferential Dense Primal Determined"*:

**Définition 6** (Fonctions SDPD) *Soit $(X, \theta)$ un espace localement convexe. On dit qu'une fonction $f : X \to \mathbb{R}_\infty$ est* Subdifferential Dense Primal Determined *(SDPD) si elle est*

*$\tau(X^*, X^{**})$-épi-pointée et si l'ensemble des $x^* \in \mathrm{Cont}[f, \tau(X^*, X^{**})]$ satisfaisant l'équation*

$$\partial f^*(x^*) = \overline{\mathrm{co}}^{w^{**}} \left[(\partial f)^{-1}(x^*)\right] \tag{5}$$

*est $\tau(X^*, X^{**})$-dense dans $\mathrm{Cont}[f^*, \tau(X^*, X^{**})]$.*

Dans cette définition, on utilise la topologie de Mackey $\tau(X^*, X^{**})$ pour trois raisons: La première est que, entre toutes les topologies présentées ci-dessus, celle-ci est la plus fine sur $X^*$, donc il y a plus de fonctions $\tau(X^*, X^{**})$-épi-pointées; la deuxième est que dans le contexte des espaces de Banach, $\tau(X^*, X^{**})$ coïncide avec la topologie induite par la norme duale de $X^*$; et finalement, la troisième est que la Proposition 3 garantit que le sous-différentiel d'une fonction conjuguée $f^*$ est semi-continue extérieurement sur $\mathrm{Cont}[f^*, \tau(X^*, X^{**})]$. Par contre, il y a une difficulté avec ce choix: la $\tau(X^*, X^{**})$-densité des points qui doivent vérifier l'équation (5) est plus difficile à obtenir.

Avec ces deux notions ($\tau$-épi-pointée et SDPD) on peut énoncer le théorème principal du Chapitre 1:

**Théorème 7** *Soit $(X, \theta)$ un espace localement convexe, et soit $f : X \to \mathbb{R}_\infty$ une fonction SDPD. Pour n'importe quelle fonction $g : X \to \mathbb{R}_\infty$ satisfaisant la condition*

$$\partial f(x) \subseteq \partial g(x), \quad \forall x \in X,$$

*il existe une constante $c \in \mathbb{R}$ telle que*

$$\overline{\mathrm{co}}\, f = \overline{(\overline{\mathrm{co}}\, g) \Box \sigma_{\mathrm{dom}\, f^*}} + c.$$

*Si de plus $\mathrm{dom}\, g^* \subseteq \overline{\mathrm{dom}\, f^*}$, alors $\overline{\mathrm{co}}\, f$ et $\overline{\mathrm{co}}\, g$ coïncident à une constante additive près.*

La proposition suivante montre que notre théorème récupère le théorème 2 et en même temps, il peut être appliqué dans des plus généraux.

**Proposition 8** *Les assertions suivantes ont lieu:*

*(a) Soit $X$ un espace de Banach avec la RNP. Alors, toute fonction norme-épi-pointée et norme-semi-continue inférieurement définie sur $X$ est SDPD.*

*(b) Soit $X$ un espace localement convexe semi-réflexif. Alors, toute fonction $\tau(X^*, X)$-épi-pointée et $w$-semi-continue inférieurement définie sur $X$ est SDPD.*

Une question naturelle est de savoir si le théorème 7 récupère la formule d'intégration de Moreau-Rockafellar pour les fonctions épi-pointées, c'est-à-dire si toute fonction $f \in \Gamma_0(X)$ qui est $\tau(X^*, X^{**})$-épi-pointée, est aussi SDPD. Une première observation est que l'équation (5) peut être décomposée en deux parties: une dépendant seulement de la fonction elle-même, et l'autre dépendant totalement de la conjuguée de cette fonction (ou de façon équivalente, de l'enveloppe convexe fermée de la fonction):

**Proposition 9** *Pour une fonction* $f : X \to \overline{\mathbb{R}}$ *et une forme linaire continue* $x^* \in X^*$, *l'équation* (5) *est satisfaite par* $f$ *si et seulement si les conditions suivantes en* $x^*$ *ont lieu:*

*(i)* $X \cap \partial f^*(x^*) = \overline{\text{co}}\,[(\partial f)^{-1}(x^*)]$.

*(ii)* $\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}}$.

*En particulier, une fonction* $f$ *qui est* $\tau(X^*, X^{**})$*-épi-pointée est SDPD quand il existe un ensemble* $\tau(X^*, X^{**})$*-dense* $D$ *de* $\text{Cont}[f^*, \tau(X^*, X^{**})]$ *tel que* $f$ *satisfait* (i) *et* (ii) *en tout point* $x^* \in D$.

Motivé par cette proposition, on définit la classe des espaces localement convexes où le théorème 7 peut être appliqué à toutes les fonctions de $\Gamma_0(X)$ qui sont $\tau(X^*, X^{**})$-épi-pointées: On les appelle *espaces SDPD*.

**Définition 10** (Espaces SDPD) *On dit qu'un espace localement convexe* $X$ *est un espace SDPD si pour toute fonction* $f \in \Gamma_0(X)$ *telle que* $\text{Cont}[f^*, \tau(X^*, X^{**})]$ *n'est pas vide, il existe un sous-ensemble* $\tau(X^*, X^{**})$*-dense* $D$ *de* $\text{Cont}[f^*, \tau(X^*, X^{**})]$ *tel que*

$$\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}}, \tag{6}$$

*en tout point* $x^* \in D$.

Il est évident que, d'aprés la proposition 8, tous les espaces de Banach jouissant de la propriété RNP et tous les espaces localement convexes semi-réflexifs sont des espaces SDPD. On cloture le résumé du Chapitre 1 avec la question suivante: Quand est-ce qu'un espace de Banach est SDPD? Comme première réponse, on a établi la proposition suivante.

**Proposition 11** *Soit* $X$ *un espace localement convexe, soit* $f \in \Gamma_0(X)$ *une fonction* $\tau(X^*, X^{**})$*-épi-pointée, et soit* $x^* \in \text{Cont}[f^*, \tau(X^*, X^{**})]$. *On a l'équivalene:*

$$\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}} \iff (f^*)'(x^*, \cdot) \text{ est } w^*\text{-semi-continue inférieurement.}$$

*En particulier,* $X$ *est un espace SDPD si et seulement si pour toute fonction* $f \in \Gamma_0(X)$ *l'ensemble*

$$D = \{x^* \in \text{int}\,(\text{dom}\, f^*, \tau(X^*, X^{**})) \ : \ (f^*)'(x^*, \cdot) \text{ est } w^*\text{-semi-continue inférieurement}\}$$

*est* $\tau(X^*, X^{**})$*-dense dans* $\text{Cont}[f^*, \tau(X^*, X^{**})]$.

Les Chapitres 2 et 3 sont consacrés à caractériser les espaces de Banach qui sont SDPD.

## Chapitre 2: Propriétés Smooth-like

Dans le contexte des espaces de Banach, la proposition 11 précédente peut être écrite comme suit: Un espace de Banach $X$ est un espace SDPD si et seulement si pour toute fonction $f \in \Gamma_0(X)$ l'ensemble

$$D = \{x^* \in \mathrm{int}\,(\mathrm{dom}\, f^*)\ :\ (f^*)'(x^*, \cdot) \text{ est } w^*\text{-semi-continue inférieurement}\}$$

est dense dans $\mathrm{int}(\mathrm{dom}\, f^*)$. Cette caractérisation est très similaire à la définition des *espaces $w^*$-Asplund*. Rappelons que pour un espace de Banach $X$ on dit que

(a) $X$ est un *espace d'Asplund* si pour toute fonction $f$ de $\Gamma_0(X)$ l'ensemble

$$\mathfrak{D}(f) := \{x \in \mathrm{int}(\mathrm{dom}\, f)\ :\ f \text{ est Fréchet-différentiable en } x\}$$

est dense dans $\mathrm{int}(\mathrm{dom}\, f)$.

(b) $X^*$ est un *espace $w^*$-Asplund* si pour toute fonction $f$ de $\Gamma_0(X^*, w^*)$ l'ensemble

$$\mathfrak{D}(f) := \{x^* \in \mathrm{int}(\mathrm{dom}\, f)\ :\ f \text{ est Fréchet-différentiable en } x^*\}$$

est dense dans $\mathrm{int}(\mathrm{dom}\, f)$.

Rappelons que l'application $* : \Gamma_0(X) \to \Gamma_0(X^*, w^*)$ est une bijection, donc $\Gamma_0(X^*, w^*)$ coïncide avec l'ensemble des fonctions conjuguées de fonctions de $\Gamma_0(X)$. Alors, il est clair que la définition des espaces $w^*$-Asplund est semblable à celle des espaces SDPD, remplaçant la $w^*$-semi-continuité inférieure par la Fréchet-différentiabilité. De plus, on sait que les espaces d'Asplund et les espaces $w^*$-Asplund sont en dualité avec des espaces qui ont la propriété RNP:

(a) $X$ est un espace d'Asplund si et seulement si $X^*$ a la propriété RNP (voir [32]).

(b) $X$ a la propriété RNP si et seulement si $X^*$ est un espace $w^*$-Asplund (voir [6]).

Motivé par cette théorie, on introduit la notion de *propriétés Smooth-like*, qui sont une généralisation de la Fréchet-différentiabilité et qui, comme on le verra plus loin, recouvre aussi l'équation (6).

Avant de définir les propriétés smooth-like, on a besoin de formaliser la notion de propriété qui nous intéresse dans le contexte des espaces de Banach. Une propriété des espaces de Banach $(\mathcal{P})$ sera considérée comme une famille de fonctions $\{\mathcal{P}_X : D(X) \to \{0, 1\}\}$, qui est indexée par la classe d'espaces de Banach et où chaque domaine $D(X)$ dépend de l'espace-indice $X$.

Intuitivement, la famille des fonctions $\{\mathcal{P}_X\}$ est telle que pour chaque $z \in D(X)$, $\mathcal{P}_X(z) = 1$ veut dire que *la propriété* $(\mathcal{P})$ *est satisfaite en* $z$. Alors, pour bien définir une propriété $(\mathcal{P})$ on a besoin de spécifier pour chaque espace $X$: 1) le domaine $D(X)$ de $\mathcal{P}_X$; et 2) qu'est-ce $\mathcal{P}_X(z) = 1$ veut dire, normalement à travers la définition de l'équivalence

$$\mathcal{P}_X(z) = 1 \iff (\mathcal{P}) \text{ est satisfaite en } z.$$

Présentons une famille de propriétés qui généralise la Fréchet-différentiabilité des fonctions convexes et $w^*$-semi-continues inférieurement: Les propriétés *"convexes $w^*$-smooth-like"*.

**Définition 12** (Propriétés convexes $w^*$-smooth-like) *Une propriété* $(\mathcal{P})$ *d' espaces de Banach est dite être* "convexe $w^*$-smooth-like" *si pour chaque espace* $X$, $D(X) := \Gamma_0(X^*, w^*) \times X^*$ *et la fonction* $\mathcal{P}_X$ *satisfait les conditions suivantes:*

*(i)* $\mathcal{P}_X$ *est* ***locale****: Pour toute paire de fonctions* $f, g \in \Gamma_0(X, w^*)$ *et pour chaque sous-ensemble ouvert* $U \subseteq X^*$, *on a*

$$f\big|_U = g\big|_U \implies \mathcal{P}_X(f, \cdot)\big|_U = \mathcal{P}_X(f, \cdot)\big|_U.$$

*(ii)* $\mathcal{P}_X$ *est* ***w*-transitive:*** *Pour tout autre espace de Banach* $Y$ *et tout opérateur linaire borné et injectif* $T : Y \to X$ *avec image fermée, on a*

$$\mathcal{P}_X(f \circ T^*, x^*) = \mathcal{P}_Y(f, T^* x^*)$$

*pour toute fonction* $f \in \Gamma_0(Y^*, w^*)$ *et tout point* $x^* \in X^*$ *(où* $T^*$ *dénote l'opérateur adjoint de* $T$*).*

*(iii)* $\mathcal{P}_X$ *est* ***ensemble-consistante****: Pour chaque ensemble fermé et borné* $K$ *de* $X$ *on a*

*(iii.a)* $\forall x^* \in X^*, \forall t > 0$, $\mathcal{P}_X(\sigma_K, x^*) = \mathcal{P}(\sigma_K, tx^*)$.

*(iii.b)* $\forall x \in X$, $\mathcal{P}_X(\sigma_{K+x}, \cdot) = \mathcal{P}_X(\sigma_K, \cdot)$.

*(iv)* $\mathcal{P}_X$ *est* ***épigraphique****: Pour chaque fonction* $f \in \Gamma_0(X)$ *et chaque point* $x^* \in X^*$, *on a*

$$\mathcal{P}_X(f^*, x^*) = \mathcal{P}_{X \times \mathbb{R}}(\sigma_{\operatorname{epi} f}, (x^*, -1)).$$

Pour simplifier la notation, on écrira $\mathcal{P}(\cdot, \cdot)$, omettant l'indice. Aussi, dans le cas d'une fonction d'appui $\sigma_K$ (où $K$ est un ensemble convexe et fermé de $X$), on écrira quelques fois $\mathcal{P}(K, \cdot)$ à la place de $\mathcal{P}(\sigma_K, \cdot)$.

Finalement, pour une fonction $f \in \Gamma_0(X^*, w^*)$, on posera

$$\mathcal{P}[f] := \{x^* \in \operatorname{dom} f \ : \ \mathcal{P}(f, x^*) = 1\},$$

et pour $f = \sigma_K$ on écrira de même $\mathcal{P}[K]$ à la place de $\mathcal{P}[\sigma_K]$.

**Définition 13** (espaces $(\mathcal{P})$-$w^*$-structuraux) *Soit* $(\mathcal{P})$ *une propriété convexe* $w^*$*-smooth-like. Un espace de Banach* $X$ *est dit être:*

*(a)* $(\mathcal{P})$*-*$w^*$*-structural si pour toute fonction* $f \in \Gamma_0(X^*, w^*)$, *l'ensemble* $\mathcal{P}[f]$ *est dense dans* $\operatorname{int}(\operatorname{dom} f)$.

*(b)* $(\mathcal{P})$*-géométrique si pour tout ensemble convexe, fermé et borné* $K$ *de* $X$, $\mathcal{P}[K]$ *est dense dans* $X^*$.

Il est évident que la Fréchet-différentiabilité est une propriété convexe $w^*$-smooth-like, et que pour un espace de Banach $X$, son dual $X^*$ est un espace $w^*$-Asplund si et seulement si $X$ est $w^*$-structural au sens de la Fréchet-différentiabilité. Dans le Chapitre 2, on définit aussi une version primale de ce type de propriétés, c'est-à-dire, dans la ligne de généralisation des espaces d'Asplund. Mais, pour l'étude des espaces SDPD, les propriétés convexes $w^*$-smooth-like sont les plus pertinentes. Pour cette raison, on se limite ici à présenter seulement le développement des propriétés de la définition 12.

Les propositions suivantes montrent que les propriétés convexes $w^*$-smooth-like possèdent beaucoup de caractérisations en termes de différentes familles de fonctions, et jouissent aussi de certaines stabilités. Tout ce qui suit a été motivé par les réductions qu'on peut trouver dans la littérature pour les espaces $w^*$-Asplund. On renvoie le lecteur au livre de R. R. Phelps [25] qui contient une étude détaillée de ce type de caractérisations, ainsi que des résultats de stabilité pour les espaces d'Asplund et les espaces $w^*$-Asplund.

**Proposition 14** *Soit $(\mathcal{P})$ une propriété convexe $w^*$-smooth-like et soit $X$ un espace de Banach. Les assertions suivantes sont équivalentes:*

*(a) Pour toute fonction $f \in \Gamma_0(X^*, w^*)$ avec $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$, on a $\operatorname{int}(\operatorname{dom} f) \cap \mathcal{P}[f] \neq \emptyset$.*

*(b) $X$ est $(\mathcal{P})$-$w^*$-structural.*

*(c) Pour toute fonction $f : X^* \to \mathbb{R}_\infty$ qui est convexe, propre, $w^*$-semi-continue inférieurement et continue, l'ensemble $\mathcal{P}[f]$ est dense dans $\operatorname{int}(\operatorname{dom} f)$.*

*(d) Pour toute fonction $f \in \Gamma_0(X^*, w^*)$ finie sur $X^*$, $\mathcal{P}[f]$ est dense dans $X^*$.*

**Théorème 15** *Soit $(\mathcal{P})$ une propriété convexe $w^*$-smooth-like, et soit $X$ un espace de Banach. Les assertions suivantes sont équivalentes:*

*(a) Pour tout ensemble $K \subseteq X$ convexe et fermé avec $\operatorname{int}(\operatorname{dom} \sigma_K) \neq \emptyset$, on a $\mathcal{P}[\sigma_K] \cap \operatorname{int}(\operatorname{dom} \sigma_K) \neq \emptyset$.*

*(b) Pour tout ensemble $K \subseteq X$ convexe et fermé, $\mathcal{P}[\sigma_K]$ est dense dans $\operatorname{int}(\operatorname{dom} \sigma_K)$.*

*(c) $X$ est $(\mathcal{P})$-géométrique.*

*(d) Pour chaque norme équivalente $p$ sur $X^*$, l'ensemble $\mathcal{P}[p]$ est dense dans $X^*$.*

**Proposition 16** *Soit $(\mathcal{P})$ une propriété convexe $w^*$-smooth-like. Un espace de Banach est $(\mathcal{P})$-structural si et seulement si $X \times \mathbb{R}$ est $(\mathcal{P})$-géométrique.*

**Proposition 17** *Soit $(\mathcal{P})$ une propriété convexe $w^*$-smooth-like, et soient $X, Y$ deux espaces de Banach avec un opérateur linaire borné $T : Y \to X$, qui est injectif et d'image fermée. Les assertions suivantes ont lieu:*

*(a) $X$ est $(\mathcal{P})$-géométrique $\Longrightarrow$ $Y$ est $(\mathcal{P})$-géométrique.*

*(b) $X$ est $(\mathcal{P})$-$w^*$-structural $\Longrightarrow$ $Y$ est $(\mathcal{P})$-$w^*$-structural.*

*En particulier, les classes des espaces $(\mathcal{P})$-$w^*$-structuraux et des espaces $(\mathcal{P})$-géométriques sont stables pour les sous-espaces fermés et pour les isomorphismes.*

En ce qui concerne les espaces $w^*$-Asplund, la caractérisation la plus importante pour nous est la suivante: Étant donné un espace de Banach $X$, son dual $X^*$ est un espace $w^*$-Asplund si et seulement si toute norme duale équivalente sur $X^*$ a un point de Fréchet-différentiabilité. Dans le cas des propriétés convexes $w^*$-smooth-like, on a obtenu le même résultat pour les espaces de Banach séparables, en utilisant une hypothèse supplémentaire: *La règle de $w^*$-somme.*

**Définition 18** ($w^*$-Sum rule) *Soit $(\mathcal{P})$ une propriété convexe $w^*$-smooth-like. On dit que $(\mathcal{P})$ a* la règle de $w^*$-somme *si pour tout espace de Banach $X$ et toute paire de fonctions $f, g \in \Gamma_0(X^*, w^*)$ on a*

$$\mathcal{P}_X(f+g,\cdot) = \min\{\mathcal{P}_X(f,\cdot), \mathcal{P}_X(g,\cdot)\}. \tag{7}$$

Pour utiliser la règle de $w^*$-somme, quelques résultats préliminaires sont nécessaires: On a besoin de couvrir un espace par "rotation de cônes". Les deux propositions suivantes montrent qu'on peut le faire pour des espaces séparables et pour les espaces duaux d'espaces séparables (qui ne sont pas nécessairement séparables).

**Proposition 19** *Soit $X$ un espace séparable et soit $C$ une cône ouvert de $X$. Alors il existe une famille dénombrable d'isomorphismes $\{T_n : X \to X \mid n \in \mathbb{N}\}$ tels que*

$$\bigcup_{n\in\mathbb{N}} T_n^{-1}(C) = X \setminus \{0\}.$$

**Proposition 20** *Soit $X$ un espace séparable et soit $C$ un cône ouvert de l'espace dual $X^*$. Alors, il existe une famille dénombrable d'isomorphismes $\{T_n : X \to X \mid n \in \mathbb{N}\}$ tels que*

$$\bigcup_{n\in\mathbb{N}} (T_n^*)^{-1}(C) = X^* \setminus \{0\}.$$

En utilisant la proposition précédente, on peut déduire le théorème suivant, qui est notre dernière réduction pour les propriétés convexes $w^*$-smooth-like.

**Théorème 21** *Soit $X$ un espace de Banach séparable et soit $(\mathcal{P})$ une propriété convexe $w^*$-smooth-like avec la règle de $w^*$-somme. Alors, $X$ est $(\mathcal{P})$-géométrique si et seulement si pour toute norme équivalente $p$ sur $X$, il existe une forme linaire continue $x^* \in X^* \setminus \{0\}$ telle que*

$$\mathcal{P}_X(p_*, x^*) = 1,$$

*où $p_*$ désigne la norme duale sur $X^*$ associée à $p$.*

## Chapitre 3: La propriété de Radon-Nikodým de faces

Dans leur article [4], A. K. Chakrabarty, P. Shunmunagaraj et C. Zălinescu ont étudié diverses propriétés de continuité du sous-différentiel et du $\varepsilon$-sous-différentiel des fonctions convexes. Leur travail rassemble beaucoup de contributions dans la même ligne, comme [8], [15], [16] et [17]. Alors, [4] est en même temps un survey et une généralisation des contributions précédemment mentionnées.

Dans [4], l'équation qui nous intéresse, c'est-à-dire, l'équation (6), est caractérisée en termes d'une notion appelée *Hausdorff-semi-continuité supérieure.*

**Définition 22** *Soit $(T,\tau)$ un espace topologique, $(Z,\theta)$ un espace localement convexe, $M : T \rightrightarrows Z$ une multi-application et $t_0$ un point de $T$. On dit que $M$ est $\tau$-$\theta$* Hausdorff-semi-continue supérieurement *($\tau$-$\theta$-H-scs, pour simplifier) en $t_0$ si*

$$\forall V \in \mathcal{N}_Z(0),\ \exists U \in \mathcal{N}_T(t_0) \text{ tel que } M(U) \subseteq M(t_0) + V.$$

Le résultat de [4] qu'on utilisera est le suivant:

**Théorème 23** ([4, Proposition 5.2]) *Soit $f \in \Gamma_0(X^*, w^*)$ et soit $x^* \in \operatorname{int}(\operatorname{dom} f)$. On a l'équivalence*

$$\partial f(x^*) = \overline{X \cap \partial f(x^*)}^{w^{**}} \iff \eth f(0,\cdot) \text{ est } \tau_{\|\cdot\|}\text{-}w \text{ H-scs en } x^*.$$

*où $\eth f : \mathbb{R}_+ \times X^* \rightrightarrows X$ est la multi-application définie par*

$$\eth f(\varepsilon, x^*) := X \cap \partial_\varepsilon f(x^*).$$

Maintenant, on présente un des résultats principaux du Chapitre 3: Les espaces SDPD sont des espaces $(\mathcal{P})$-$w^*$-structuraux par rapport à une propriété convexe $w^*$-smooth-like:

**Théorème 24** *Soit $(\mathcal{P})$ la propriété d'espace de Banach définie comme suit: Pour chaque espace de Banach $X$, le domaine de $\mathcal{P}_X$ est $D(X) := \Gamma_0(X^*, w^*) \times X^*$, et $\mathcal{P}_X$ est définie par l'équivalence*

$$\mathcal{P}_X(f, x^*) = 1 \iff x^* \in \operatorname{int}(\operatorname{dom} f^*) \text{ et } \partial f(x^*) = \overline{X \cap \partial f(x^*)}^{w^{**}}. \tag{8}$$

*Alors, la propriété $(\mathcal{P})$ est convexe $w^*$-smooth-like et donc, d'aprés l'équivalence* (8)*, être un espace SDPD revient à être un espace $(\mathcal{P})$-$w^*$-structural.*

Maintenant, on va donner une caractérisation des espaces SDPD via les "faces exposées". Il y a beaucoup de définitions des faces d'un ensemble dans la littérature, mais on va adopter la convention que, dans un espace de Banach $X$, une face $F$ d'un ensemble $K \subseteq X$ est un *sous-ensemble exposé* de $K$, c'est-à-dire, il existe une forme linaire continue $x^* \in X^* \setminus \{0\}$ telle que

$$F = \{x \in K \ : \ \langle x^*, x\rangle = \sigma_K(x^*)\}.$$

Dans ce cas, on écrira $F := F[K, x^*]$ pour éviter toute ambiguïté, quand cela parait nécessaire.

**Définition 25** (Face $\theta$-exposée) *Soit $K \subseteq X$ un ensemble convexe et fermé, et soit $F$ une face de $K$. Soit aussi $\theta$ une topologie localement convexe entre $w(X, X^*)$ et $\tau_{\|\cdot\|}$ (où $\tau_{\|\cdot\|}$ désigne la topologie sur $X$ induite par la norme $\|\cdot\|$). On dit que $F$ est une* face $\theta$-exposée *de $K$ par une fonctionnelle $x^* \in X^* \setminus \{0\}$ si $F = F[K, x^*]$ et pour tout $\theta$-voisinage $V$ de $0$, il existe $\alpha > 0$ tel que*

$$S(K, x^*, \alpha) \subseteq F + V,$$

*où $S(K, x^*, \alpha) := \{x \in K \ : \ \langle x^*, x\rangle > \sigma_K(x^*) - \alpha\}$ est la tranche de $K$ induite par $x^*$ et $\alpha$. Dans ce cas, on dit que $x^*$ $\theta$-expose $F$. L'ensemble des fonctionnelles de $X^*$ qui $\theta$-exposent une face de $K$, sera noté $E[K, \theta]$.*

**Définition 26** (Propriété de Radon-Nikodým de faces) *Un espace de Banach $X$ a la* propriété de Radon-Nikodým de faces *(ou est* FRNP*, par acronyme en Anglais "Faces Radon-Nikodým Property), si pour tout ensemble $K$ de $X$ qui est convexe, fermé et borné, $E[K, w]$ est dense dans $X^*$.*

**Proposition 27** *Soit $(\mathcal{P})$ la propriété convexe $w^*$-smooth-like considérée dans le théorème 24. Pour tout espace de Banach $X$ et tout ensemble $K$ convexe, fermé et borné de $X$, on a $\mathcal{P}_X[K] = E[K, w]$.*

*Alors, un espace a la FRNP si et seulement si il est $(\mathcal{P})$-géométrique.*

À partir de ce qui a été déjà obtenu, on déduit le corollaire suivant:

**Corollaire 28** *Si un espace de Banach $X$ a la propriété FRNP, alors tout ensemble $K$ convexe, fermé et borné de $X$ peut être récupéré comme l'enveloppe convexe fermée de ses faces $w$-exposées, c'est-à-dire*

$$K = \overline{\mathrm{co}}\left[\bigcup_{x^* \in E[K,w]} F[K, x^*]\right].$$

Suite au théorème 23, on a étudié aussi une version forte des espaces SDPD et de la FRNP, basée sur la propriété des espaces de Banach $(s\mathcal{P})$, définie comme suit: Pour chaque espace de Banach $X$, le domaine de $s\mathcal{P}_X$ est $D(X) := \Gamma_0(X^*, w^*) \times X^*$ et la fonction $s\mathcal{P}_X$ est donnée par l'équivalence

$$s\mathcal{P}_X(f, x^*) = 1 \iff x^* \in \operatorname{int}(\operatorname{dom} f) \text{ et } \eth f(0, \cdot) \text{ est } \tau_{\|\cdot\|}\text{-}\tau_{\|\cdot\|} \text{ H-scs en } x^*. \tag{9}$$

**Proposition 29** *La propriété $(s\mathcal{P})$ est une propriété convexe $w^*$-smooth-like. De plus, la propriété $(s\mathcal{P})$ vérifie la règle de $w^*$-somme.*

**Définition 30** *Étant donné un espace de Banach $X$, on dit:*

1. *$X$ est un **espace strong-SDPD** s'il est $(s\mathcal{P})$-$w^*$-structural.*

2. *$X$ a la **strong-FRNP** s'il est $(s\mathcal{P})$-géométrique.*

Pour la propriété $(s\mathcal{P})$ on peut obtenir un résultat similaire au Corolaire 28, mais plus fort, compte tenu du fait que $(s\mathcal{P})$ a la règle de $w^*$-somme:

**Proposition 31** *Si un espace de Banach $X$ a la strong-FRNP, alors tout ensemble $K$ convexe, fermé et borné de $X$ peut être récupéré comme l'enveloppe convexe fermée de ses faces $\tau_{\|\cdot\|}$-exposées, c'est-à-dire*

$$K = \overline{\text{co}} \left[ \bigcup_{x^* \in E[K, \tau_{\|\cdot\|}]} F[K, x^*] \right].$$

*De plus, si $X$ est séparable, la condition ci-dessus est aussi suffisante.*

Dans la dernière partie du Chapitre 3, on compare les propriétés suivantes: La RNP, la strong-FRNP et la FRNP. La proposition suivante résume nos résultats.

**Proposition 32** *Soit $X$ un espace de Banach. Les implications suivantes ont lieu:*

$$X \textit{ a la RNP} \implies X \textit{ a la strong-FRNP} \implies X \textit{ a la FRNP}.$$

*De plus, si $X$ possède une copie de $c_0$, alors $X$ n'a pas la propriété FRNP.*

Maintenant, nous ne savons pas si la FRNP est équivalente ou non à la RNP. Cette question est encore ouverte. On espère pouvoir y répondre dans le futur.

# Partie II: Géométrie Non-convexe

La deuxième partie de la thèse correspond à l'étude de la différentiabilité de la projection métrique sur des sous-ensembles des espaces de Hilbert. Ce problème est motivé par trois travaux différents: L'article de 1973 de R. B. Holmes [19]; l'article de 1982 de S. Fitzpatrick et R. R. Phelps [14]; et l'article de 1984 de J.-B. Poly et G. Raby [27].

Les deux premiers articles ci-dessus cités ont examiné la différentiabilité de la projection métrique $P_K$, quand $K$ est un corps convexe (i.e. $\operatorname{int} K \neq \emptyset$) d'un espace de Hilbert $X$ (doté de la norme hilbertienne $\|\cdot\|$ associée au produit scalaire $\langle \cdot, \cdot \rangle$). Ces deux articles, [19] et [14], ont établi des résultats qui relient la différentiabilité de $P_K$ à la lissité de la frontière de $K$, $\operatorname{bd} K$, au sens de variétés différentiables. Rappelons les théorèmes principaux de ces articles:

**Théorème 33** (Holmes, 1973) *Soit $X$ un espace de Hilbert, $K$ un corps convexe de $X$ et $x_0 \in \operatorname{bd} K$. Supposons que $\operatorname{bd} K$ soit une sous-variété de $X$ en $x_0$ de classe $\mathcal{C}^{p+1}$ (avec $p \geq 1$). Définissons le rayon normal ouvert de $K$ en $x_0$ comme l'ensemble*

$$\operatorname{Ray}_{x_0}(K) := \{x_0 + t\hat{n}(x_0) \ : \ t > 0\},$$

*où $\hat{n}(x_0)$ désigne le vecteur normal extérieur de $K$ en $x_0$. Alors, il existe un voisinage $W$ de $\operatorname{Ray}_{x_0}(K)$ tel que $d_K$ soit de classe $\mathcal{C}^{p+1}$ sur $W$ et $P_K$ de classe $\mathcal{C}^p$ sur $W$.*

La contribution fondamentale de [14] (Théorème 34 ci-dessous) est d'avoir pu identifier la condition à ajouter dans l'énoncé du théorème précédent pour aboutir à une caractérisation pour que $P_K$ soit de classe $\mathcal{C}^p$ sur un voisinage du rayon ouvert ci-dessus. Le résultat peut être énoncé sous la forme d'équivalence suivante:

**Théorème 34** (Fitzpatrick-Phelps, 1982) *Soit $X$ un espace de Hilbert et $K$ un corps convexe de $X$. Alors, $\operatorname{bd} K$ est une sous-variété de classe $\mathcal{C}^{p+1}$ (avec $p \geq 1$) de $X$ si et seulement si $P_K$ est de classe $\mathcal{C}^p$ sur $X \setminus K$ et, pour tout $x \in X \setminus K$, la restriction de $DP_K(x)$ à l'hyperplan*

$$H[x] := \{z \in X \ : \ \langle z, x - P_K(x) \rangle = 0\}$$

*est inversible.*

De façon indépendante, le troisième article [27] a étudié la différentiabilité de la projection métrique $P_M$ quand l'ensemble $M$ est lui-même une sous-variété de classe $\mathcal{C}^{p+1}$ d'un espace euclidien de dimension finie.

**Théorème 35** (Poly-Raby, 1984) *Soit $M$ un sous-ensemble d'un espace euclidien $X$ de dimension finie, et soit $m_0 \in M$. L'ensemble $M$ est une sous-variété de $X$ en $m_0$ de classe $\mathcal{C}^{p+1}$ (avec $p \geq 1$) si et seulement si la fonction distance carré $d_M^2(\cdot)$ est de classe $\mathcal{C}^{p+1}$ dans un voisinage de $m_0$.*

On remarque que ces trois théorèmes considèrent des ensembles qui sont au moins des sous-variétés de classe $\mathcal{C}^2$. En fait, en dehors de cette classe, il y a beaucoup de contre-exemples même en dimension finie (voir, par exemple, [20] et [14]).

La contribution de cette seconde partie de la thèse est de relaxer les hypothèses de ces théorèmes: Pour les théorèmes 33 et 34, on remplace l'hypothèse de convexité par une autre plus générale: la *prox-régularité*. La projection métrique sur un ensemble prox-régulier étant bien définie (comme application) "proche" de cet ensemble. On a établi, pour des ensembles prox-réguliers, des résultats similaires aux théorèmes 34 et 35. Ces résultats sont obtenus sous des formes quantifiées faisant intervenir les constantes de prox-régularité des ensembles. Pour ce qui concerne le théorème 35, on étend (partiellement) le résultat aux espaces de Hilbert de dimension arbitraire (finie ou infinie) et on donne un voisinage quantifié où la projection métrique est différentiable.

## Chapitre 4: Sous-variétés et prox-régularité

Ce chapitre est pour une bonne partie un résumé de tous les éléments qui sont nécessaires pour développer notre contribution dans le chapitre 5. Les rappels nécessaires concernent: Les sous-variétés d'un espace de Hilbert, la théorie du calcul proximal, et finalement les ensembles prox-réguliers. Nous ne donnerons dans le résumé ici que les définitions et propositions qui sont fondamentales pour comprendre les théorèmes que nous énonçons ci-dessous du chapitre 5.

Dans ce qui suit, $X$ sera toujours un espace de Hilbert avec $\|\cdot\|$ sa norme euclidienne et $\langle\cdot,\cdot\rangle$ son produit scalaire, et $p$ sera toujours un entier supérieur ou égal à 1.

Les premières notions à rappeler sont celles de sous-variétés et d'espaces tangents.

**Définition 36** (Sous-variétés de classe $\mathcal{C}^p$) *Un sous-ensemble $M$ de $X$ est dit être une $\mathcal{C}^p$-sous-variété en un point $m_0 \in M$ ou une sous-variété en $m_0 \in M$ de classe $\mathcal{C}^p$ s'il existe un voisinage ouvert $U$ de $m_0$, un sous-espace fermé $Z$ de $X$ (qui s'appelle l'espace modèle) et une application $\varphi : U \to \varphi(U) \subseteq X$ tels que*

1. *$\varphi$ est un $\mathcal{C}^p$-difféomorphisme, c'est-à-dire, $\varphi(U)$ est un sous-ensemble ouvert de $X$, $\varphi : U \to \varphi(U)$ est bijective, et $\varphi, \varphi^{-1}$ sont toutes deux de classe $\mathcal{C}^p$.*
2. *$\varphi(m_0) = 0$ et $\varphi(M \cap U) = Z \cap \varphi(U)$.*

*On dit simplement que $M$ est une $\mathcal{C}^p$-sous-variété de $X$ s'il est une $\mathcal{C}^p$-sous-variété en tout point $m$ de $M$, avec le même espace modèle $Z$.*

**Définition 37** (Espace tangent) *Soit $M \subseteq X$ une sous-variété en un point $m_0 \in M$ de classe $\mathcal{C}^p$. On définit* l'espace tangent *de $M$ en $m_0$ comme l'ensemble*

$$T_{m_0}M := \{h \in X \ : \ \exists \gamma : ]-1,1[\to M,\ \mathcal{C}^1\text{-courbe avec } \gamma(0) = m_0 \text{ et } \gamma'(0) = h\}.$$

La prochaine proposition nous permet de représenter les sous-variétés localement comme le graphe d'une application différentiable.

**Proposition 38** *Soit $M$ un sous-ensemble de $X$ et $m_0 \in M$. Alors $M$ est une sous-variété en $m_0$ de classe $\mathcal{C}^p$ si et seulement s'il existe un sous-espace fermé $Z$ de $X$, un voisinage ouvert $U$ de $m_0$ dans $X$, un voisinage ouvert $V$ de $0$ dans $Z$ et une application $\theta : V \to Z^\perp$ de classe $\mathcal{C}^p$ tels que $\theta(0) = 0$, $D\theta(0) = 0$ et*

$$M \cap U = \big(L^{-1}(\operatorname{gph}\theta) + m_0\big) \cap U,$$

*où $L : X \to Z \times Z^\perp$ est l'isomorphisme canonique donné par $L(x) = (\pi_Z(x), \pi_{Z^\perp}(x))$, avec $\pi_Z$ et $\pi_{Z^\perp}$ les projections parallèles associées à la décomposition $X = Z \oplus Z^\perp$. Dans ce cas, on a $Z = T_{m_0}M$.*

Dans le cadre convexe, on sait que pour tout ensemble convexe fermé, la projection métrique existe toujours et elle est unique. En dehors de ce contexte, c'est nécessaire de mieux formaliser ce que l'on entend par "projection métrique". Dans ce qui suit, pour un ensemble non-vide $S$ de $X$, on désignera par $d_S(\cdot)$ (ou $d(\cdot; S)$ si besoin est) la fonction distance de $S$, c'est-à-dire, pour $x \in X$

$$d_S(x) := \inf\{\|x - s\| \ : \ s \in S\}.$$

**Définition 39** (Projection métrique) *Soit $S$ un ensemble non-vide de $X$ et soit $x \in X$ un point fixé. On dit qu'un point $s$ de $S$ est une* projection de $x$ sur $S$ *si*

$$\|x - s\| = d_S(x).$$

*L'ensemble de toutes les projections de $x$ sur $S$ sera notée (comme il est usuel)* $\operatorname{Proj}_S(x)$ *ou* $\operatorname{Proj}(x; S)$. *Si* $\operatorname{Proj}_S(x)$ *est un singleton, alors son unique point s'appelle* la projection métrique de $x$ sur $S$ *et elle est notée $P_S(x)$.*

Via la définition 39, on peut introduire l'un des outils sur lesquels est basée notre contribution: Le *cône normal proximal.* Cet outil est l'une des notions fondamentales de la théorie du calcul proximal. Quand l'on est en dehors du contexte convexe, on a besoin d'une notion appropriée de *cône normal.* Bien qu'il y ait plusieurs concepts de cônes normaux dans la théorie de l'analyse variationnelle, le mieux quand on travaille avec les projections est le cône normal proximal, qui est défini avec cet objectif. Ici, on ne fait que rappeler cette notion, et on renvoie le lecteur au livre de Clarke, Ledyaev, Stern et Wolenski [5] pour les diverses propriétés et plus de détails.

**Définition 40** (Cône normal proximal) *Soit $S$ un sous-ensemble non-vide de $X$ et $\bar{s} \in S$. On dit qu'un vecteur $\zeta \in X$ est un* vecteur normal proximal *de $S$ en $\bar{s}$ s'il existe $t > 0$ tel que*

$$\bar{s} \in \operatorname{Proj}_S(\bar{s} + t\zeta).$$

*L'ensemble de tous les vecteurs normaux proximaux de $S$ en $\bar{s}$ s'appelle le* cône normal proximal *de $S$ en $\bar{s}$, et il est désigné par $N^P(S; \bar{s})$.*

A partir du cône normal proximal, on définit la notion de *rayon normal*, qui a été utilisé dans le théorème 33: Pour un ensemble fermé $S$ de $X$, un point $x \in \operatorname{bd} S$ et un réel $\lambda > 0$, quand le cône normal proximal de $S$ en $x$ est de la forme

$$N^P(S;x) = \{t\nu \ : \ t > 0\},$$

pour un certain vecteur $\nu$ avec $\|\nu\| = 1$, on définit le *rayon normal ouvert* de $S$ en $x$ et le *rayon normal ouvert $\lambda$-tronqué* de $S$ en $x$ comme les ensembles

$$\mathrm{Ray}_x(S) := \{x + t\nu \ : \ t \in ]0, +\infty[\,\} \tag{10}$$

$$\mathrm{Ray}_{x,\lambda}(S) := \{x + t\nu \ : \ t \in ]0, \lambda[\,\} \tag{11}$$

respectivement. Nous verrons plus loin que notre extension du théorème 33 fait intervenir, au lieu du rayon normal ouvert, le rayon normal ouvert tronqué avec un réel $\lambda > 0$ approprié.

Maintenant, on présente la dernière notion fondamentale dont nous avons besoin: Les ensembles prox-réguliers. Avant de donner la définition, on veut fixer certaines notations. Pour un point $x \in X$ et un réel $\alpha > 0$, on note par $B_X(x, \alpha)$ la boule ouverte centrée en $x$ et de rayon $\alpha$, et par $B_X[x, \alpha]$ la boule fermée de mêmes paramètres. Aussi, on écrit $\mathbb{B}_X$ pour désigner la boule unitaire fermée de $X$ centrée à l'origine, c'est-à-dire, $\mathbb{B}_X := B_X[0, 1]$.

**Définition 41** (Ensemble prox-régulier) *Étant donnés un réel étendu $r \in ]0, +\infty]$ et un réel $\alpha > 0$, on dit qu'un ensemble fermé $S$ de $X$ est $(r, \alpha)$-*prox-régulier en *$x_0 \in S$ si pour tout $x \in S \cap B_X(x_0, \alpha)$ et tout $\zeta \in N^P(S; x) \cap \mathbb{B}_X$, on a*

$$x \in \mathrm{Proj}_S(x + t\zeta), \quad \textit{pour tout réel } t \in [0, r]. \tag{12}$$

*On dit que $S$ est $r$-*prox-régulier en *$x_0$ s'il est $(r, \alpha)$-prox-régulier en $x_0$ pour un certain $\alpha > 0$, et on dit simplement que $S$ est *prox-régulier en *$x_0$, s'il est $r$-prox-régulier en $x_0$ pour un réel étendu $r \in ]0, +\infty]$.*

*Par conséquence, on dit que $S$ est $r$-prox-régulier (resp. prox-régulier) s'il est $r$-prox-régulier (resp. prox-régulier) en tout point $x \in S$.*

Il est évident à partir de la définition que si $S$ est $(r, \alpha)$-prox-régulier en $x_0$, alors il est aussi $(r', \alpha')$-prox-régulier en $x_0$, pour tout $r' \in ]0, r]$ et tout $\alpha' \in ]0, \alpha]$.

Il y a beaucoup de travaux à propos des ensembles prox-réguliers, mais on renvoie le lecteur au survey très approfondi et détaillé de G. Colombo et L. Thibault [7] sur le sujet.

Dans la thèse [22] de M. Mazade il y a des caractérisations quantifiées de la prox-régularité locale. Ces caractérisations ont été motivées par le célèbre article de R. A. Poliquin, R. T. Rockafellar et L. Thibault [26], de l'année 2000. Nous rappelons ci-dessous les versions dans [22], qui vont nous permettre d'avoir des résultats en termes de voisinages quantifiés des rayons normaux. Ces résultats sont basés sur deux ensembles apparaissant dans [22].

Pour $r \in ]0,+\infty]$ et $\alpha > 0$, on considère les élargissements locaux suivants de $S$ en un point $x_0 \in S$:

$$\mathcal{R}_S(x_0,r,\alpha) := \left\{x + tv \ : \ x \in S \cap B_X(x_0,\alpha),\ t \in [0,r[,\ v \in N^P(S;x) \cap \mathbb{B}_X\right\}, \tag{13}$$
$$\mathcal{W}_S(x_0,r,\alpha) := \{u \in X \ : \ \mathrm{Proj}_S(u) \cap B_X(x_0,\alpha) \neq \emptyset,\ d_S(u) < r\}. \tag{14}$$

On résume les résultats provenant de [22] dans le théorème suivant:

**Théorème 42** (Voir [22], 2011) *Soit $S$ un ensemble fermé de $X$, $x_0 \in S$, $r \in ]0,+\infty]$ et $\alpha > 0$. Les assertions suivantes sont équivalentes:*

*(i) $S$ est $(r,\alpha)$-prox-régulier en $x_0$;*

*(ii) L'ensemble $\mathcal{W}_S(x_0,r,\alpha)$ est ouvert et $P_S$ est bien définie et localement Lipschitz-continue en $\mathcal{W}_S(x_0,r,\alpha)$;*

*(iii) L'ensemble $\mathcal{W}_S(x_0,r,\alpha)$ est ouvert et $d_S$ est continument différentiable sur $\mathcal{W}_S(x_0,r,\alpha)\setminus S$, avec $\nabla d_S(u) = \frac{u-P_S(u)}{d_S(u)}$ pour tout point $u \in \mathcal{W}_S(x_0,r,\alpha)$;*

*(iv) Pour chaque point $x \in S \cap B(x_0,\alpha)$ et chaque $\zeta \in N^P(S;x)$ on a*

$$\langle \zeta, x' - x\rangle \le \frac{\|\zeta\|}{2r}\|x'-x\|^2 \quad \textit{pour tout } x' \in S.$$

*En plus, si $S$ est $(r,\alpha)$-prox-régulier en $x_0$, alors les ensembles $\mathcal{R}_S(x_0,r,\alpha)$ et $\mathcal{W}_S(x_0,r,\alpha)$ coïncident, et pour chaque $\gamma \in ]0,1[$ l'application $P_S(\cdot)$ est Lipschitz-continue sur $\mathcal{W}_S(x_0,\gamma r,\alpha)$ avec $(1-\gamma)^{-1}$ comme constante de Lipschitz.*

Finalement, on présente la notion de corps fermé, qui remplace celui de corps convexe dans notre contexte.

**Définition 43** (Corps fermé) *Un ensemble $S$ de $X$ est dit être* corps fermé *(relatif à $X$) autour de $x_0 \in S$ s'il existe un voisinage ouvert $U$ de $x_0$ tel que $U \cap S = U \cap \left(\overline{\mathrm{int}\, S}\right)$ et $U \cap \mathrm{int}\, S$ est connexe.*

*Si $U = X$, c'est-à-dire, $S = \overline{\mathrm{int}\, S}$ et $\mathrm{int}\, S$ est connexe, on dit simplement que $S$ est un corps fermé (relatif à $X$).*

Rappelons qu'un ensemble $S$ de $X$ est dit être *épi-Lipschitzien* en $x_0 \in S$ dans la direction $h \in X \setminus \{0\}$ s'il existe un voisinage $U$ de $x_0$, un sous-espace supplémentaire topologique $Z$ de $\mathbb{R}h$, et une fonction $f : Z \to \mathbb{R}$ Lipschitz-continue tels que, écrivant $X = Z \oplus \mathbb{R}h$, on a

$$U \cap S = \{z + rh \in U \ : \ (z,r) \in \mathrm{epi}\, f\}. \tag{15}$$

La proposition suivante montre que les ensembles avec une frontière $\mathcal{C}^{p+1}$-lisse (au sens que cette frontière est une sous-variété de classe $\mathcal{C}^{p+1}$) sont prox-réguliers et aussi sont représentables comme graphe/épigraphe d'une fonction Lipschitzienne. Cette proposition ne se trouve pas dans la thèse de façon explicite, mais on l'énonce ici pour montrer que la prox-régularité et la propriété épi-Lipschitz sont en fait des conditions nécessaires, quand on a des frontières de classe $\mathcal{C}^2$ ou plus.

**Proposition 44** *Les assertions suivantes ont lieu:*

*(a) Soit $M \subseteq X$ une sous-variété en $m_0 \in M$ de classe $\mathcal{C}^2$. Alors, $M$ est prox-régulier en $m_0$ et*

$$N^P(M; m_0) = (T_{m_0}M)^{\perp}.$$

*(b) Soit $S \subseteq X$ un corps fermé autour de $x_0 \in \operatorname{bd} S$. Supposons que* $\operatorname{bd} S$ *soit une sous-variéte en $x_0$ de classe $\mathcal{C}^2$. Alors, $S$ est prox-régulier et épi-Lipschitzien en $x_0$ et il existe un vecteur $\nu \in X$ avec $\|\nu\| = 1$ tel que*

$$N^P(S; x_0) = \{t\nu \ : \ t > 0\}.$$

## Chapitre 5: Différentiabilité de la projection métrique

Ici on résume les résultats principaux de la seconde partie de la thèse.

**Théorème 45** *Soit $S \subseteq X$ un corps fermé autour de $x_0 \in \operatorname{bd} S$, et soit un entier $p \geq 1$. Supposons qu'il existe $r \in ]0, +\infty]$ et $\alpha > 0$ tels que $B_X(x_0, \alpha) \cap \operatorname{bd} S$ soit une sous-variété de classe $\mathcal{C}^{p+1}$ et que $S$ soit $r$-prox-régulier en $x_0$. Alors, il existe un voisinage $V$ de* $\mathrm{Ray}_{x_0,r}(S)$ *tel que*

- *$d_S$ est de classe $\mathcal{C}^{p+1}$ en $V$;*
- *$P_S$ est de classe $\mathcal{C}^p$ en $V$.*

*En plus, si $S$ est $(r, \alpha)$-prox-régulier en $x_0$, alors*

*$d_S$ est de classe $\mathcal{C}^{p+1}$ sur $\mathcal{W}_S(x_0, r, \alpha) \setminus S$.*

*$P_S$ est de classe $\mathcal{C}^p$ sur $\mathcal{W}_S(x_0, r, \alpha) \setminus S$.*

Pour un ensemble $S$ de $X$ et un point $x \in X$ tel que $P_S(x)$ existe, on définit l'hyperplan $H[x]$ comme

$$H[x] := \{z \in X \ : \ \langle z, x - P_S(x) \rangle = 0\}.$$

**Théorème 46** *Soit $S$ un corps fermé de $X$ et soit $x_0 \in \operatorname{bd} S$. Supposons que $S$ soit $r$-prox-régulier et épi-Lipschitzien en $x_0$. Alors, les assertions suivantes sont équivalentes:*

*(i)* $\operatorname{bd} S$ *est une sous-variété en $x_0$ de classe $\mathcal{C}^{p+1}$;*

*(ii) Il existe $\alpha > 0$ tel que $P_S$ soit de classe $\mathcal{C}^p$ sur $\mathcal{W}_S(x_0, r, \alpha) \setminus S$ et pour chaque $u \in \mathcal{W}_S(x_0, r, \alpha) \setminus S$, la restriction de $DP_S(u)$ à $H[u]$ est inversible comme application de $H[u]$ dans $H[u]$;*

*(iii) il exite un voisinage $U$ de $x_0$ tel que $P_S$ soit de classe $\mathcal{C}^p$ sur $U \setminus S$ et tel que, pour tout $u \in U \setminus S$, $DP_S(u)$ soit surjective de $H[u]$ dans $H[u]$.*

Les théorèmes 45 et 46 sont des extensions significatives des théorèmes 33 et 34, respectivement. Notons que ces théorèmes donnent en plus une quantification en termes de la constante de prox-régularité. Aussi, dans le théorème 46, les hypothèses de prox-régularité et de la propriété épi-Lipschitz sont en même temps nécessaires.

Au niveau des sous-variétés, on a obtenu un résultat similaire au théorème 45, qui généralise et quantifie la nécessité du théorème 35.

**Théorème 47** *Soit $M$ un ensemble fermé de $X$ qui est une sous-variété en $m_0 \in M$ de classe $\mathcal{C}^{p+1}$, avec $p \geq 1$. Si $M$ est $r$-prox-régulier en $m_0$, alors il existe $\alpha > 0$ tel que*

- *$d_M^2(\cdot)$ est de classe $\mathcal{C}^{p+1}$ sur $\mathcal{W}_M(m_0, r, \alpha)$;*
- *$P_M$ est de classe $\mathcal{C}^p$ sur $\mathcal{W}_M(m_0, r, \alpha)$.*

Le problème fondamental en connexion avec ce dernier théorème est sa réciproque. On est arrivé à une réciproque partielle, qui est basée sur un renforcement du concept de sous-variété de classe $\mathcal{C}^{p+1}$.

Rappelons maintenant la notion de locale uniforme continuité. Soit $E$ et $F$ deux espaces de Banach et $U$ un sous-ensemble ouvert de $E$. On écrira $\mathcal{C}^{k,0}_{\text{loc}}(U; F)$ (avec $k \geq 0$) l'ensemble de toutes les applications $f : U \to F$ qui sont de classe $\mathcal{C}^k$ et telles que la $k$-ième dérivée est *localement uniformément continue*: pour chaque $u \in U$ il existe $\delta_0 > 0$ avec $B_X(u, \delta_0) \subseteq U$ tel que $D^k f$ soit uniformément continu sur $B_X(u, \delta_0)$, c'est-à-dire,

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall u, u' \in B_X(u_0, \delta_0),\ \|u - u'\| \leq \delta \Rightarrow \|D^k f(u) - D^k f(u')\| \leq \varepsilon.$$

En même temps, on désignera par $\mathcal{C}^{k,1}_{\text{loc}}(U; F)$ l'ensemble de toutes les applications $f : U \to F$ qui sont de classe $\mathcal{C}^k$ et telles que la $k$-ième dérivée est *localement Lipschitz continue.*

D'après la Proposition 38 un ensemble $M$ de $X$ est une sous-variété en $m_0 \in M$ de classe $\mathcal{C}^p$ s'il existe un sous-espace vectoriel fermé $Z$ de $X$, un voisinage $U$ de $m_0$ dans $X$, un voisinage $V_Z$ de 0 dans $Z$, et une application $\theta : V_Z \to Z^\perp$ tels que

(i) $\theta$ est de classe $\mathcal{C}^p$;

(ii) $\theta(0) = 0$ et $D\theta(0) = 0$; et

(iii) $M \cap U = (L^{-1}(\operatorname{gph} \theta) + m_0) \cap U$, où $L : X \to Z \times Z^\perp$ est l'isomorphisme canonique donné par $L(x) = (\pi_Z(x), \pi_{Z^\perp}(x))$ (avec $\pi_Z$, $\pi_{Z^\perp}$ les projections parallèles associées à la décomposition $X = Z \oplus Z^{\perp}$).

Maintenant, on introduit deux renforcements de la notion de sous-variétés: Si en plus, $D^p\theta$ est localement uniformément continue autour de 0, on dira que $M$ est une $\mathcal{C}^{p,0}$-*sous-variété en* $m_0$, et si $D^p\theta$ est localement Lipschitz continue autour de 0, on dira que $M$ est une $\mathcal{C}^{p,1}$-*sous-variété en* $m_0$. On a l'impression que ces notions doivent être connues dans la littérature, mais malheureusement on n'a pas trouvé de références.

**Lemme 48** *Soient $X$, $Y$ et $Z$ trois espaces de Hilbert et soit $U$ un voisinage ouvert de 0 dans $X$. Considérons une application continue $T : U \to \mathcal{L}(Y;Z)$ (où $\mathcal{L}(Y;Z)$ désigne l'espace des opérateurs linaires continus de $Y$ dans $Z$) et un entier $p \geq 1$. Définissons l'application $g : U \times Y \to Z$ donnée par*

$$g(u,y) := T(u)y.$$

*Alors, $T$ est de classe $\mathcal{C}^p$, quand $g$ est de classe $\mathcal{C}^p$, et il existe un voisinage $V$ de 0 dans $Y$ tel que la famille $\{D^p g(\cdot,v)\}_{v\in V}$ est* localement equi-uniformément continue*, c'est-à-dire, pour chaque $u_0 \in U$, il existe $\delta_0 > 0$ avec $B_X(u_0,\delta_0) \subseteq U$, tel que pour chaque $\varepsilon > 0$*

$$\exists \delta > 0,\ \forall u,u' \in B_X(u_0,\delta_0),\ \|u-u'\| \leq \delta \ \Rightarrow \sup_{v\in V} \|D^p g(u,v) - D^p g(u',v)\| \leq \varepsilon. \qquad (16)$$

*En plus, si $g\big|_V \in \mathcal{C}^{p,0}_{\text{loc}}(U \times V;Z)$ (respectivement, $g\big|_V \in \mathcal{C}^{p,1}(U \times V;Z)$), alors $T \in \mathcal{C}^{p,0}_{\text{loc}}(U;\mathcal{L}(Y;Z))$ (respectivement $T \in \mathcal{C}^{p,1}(U;\mathcal{L}(Y;Z))$).*

Avec ce lemme fondamental, on a obtenu une réciproque partielle du théorème 47. On caractérise les $\mathcal{C}^{p,0}$-sous-variétés et les $\mathcal{C}^{p,1}$-sous-variétés en termes de la différentiabilité de la fonction distance carré.

**Théorème 49** *Soit $M$ un sous-ensemble fermé de $X$ et soit $m_0 \in M$. Les assertions suivantes ont lieu:*

*(a) $M$ est une $\mathcal{C}^{p+1,0}$-sous-variété en $m_0$ si et seulement si il existe un voisinage $U$ de $m_0$ tel que*

*(a.i) $d_M^2(\cdot)$ est de classe $\mathcal{C}^{p+1,0}$ sur $U$;*

*(a.ii) $P_M$ est bien définie sur $U$ et elle appartient à $\mathcal{C}^{p,0}_{\text{loc}}(U;X)$.*

*(b) $M$ est une $\mathcal{C}^{p+1,1}$-sous-variété en $m_0$ si et seulement si il existe un voisinage $U$ de $m_0$ tel que*

*(b.i) $d_M^2(\cdot)$ est de classe $\mathcal{C}^{p+1,1}$ sur $U$;*

*(b.ii) $P_M$ est bien définie sur $U$ et elle appartient à $\mathcal{C}^{p,1}_{\text{loc}}(U;X)$.*

Ce qui reste est de savoir si l'on peut trouver une réciproque du théorème 47 pour les $\mathcal{C}^{p+1}$-sous-variétés sans aucune condition de continuité uniforme de la dernière dérivée de la fonction distance carré. Une possibilité est de vérifier si toute fonction distance carré $d_M^2(\cdot)$ satisfait la condition d'equi-continuité uniforme du Lemme 48. Il est aussi possible que la stratégie soit différente pour montrer la suffisance dans le théorème 47. Cette question est ouverte.

# Bibliographie - Introduction Générale


[1] J. Benoist and A. Daniilidis, "Integration of Fenchel subdifferentials of epi-pointed functions", *SIAM Journal on Optimization*, vol. 12, no. 3, pp. 575–582, 2002.

[2] R. Bourgin, *Geometric Aspects of Convex Sets with the Radon-Nikodým Property*. New York: Springer-Verlag, 1983.

[3] R. Burachik, J. E. Martinez-Legaz, and M. Rocco, "On a sufficient condition for equality of two maximal monotone operators", *Set-Valued Analysis*, vol. 18, no. 3-4, pp. 327–335, 2010.

[4] A. K. Chakrabarty, P. Shunmunagaraj, and C. Zalinescu, "Continuity properties for the subdifferential and $\varepsilon$-subdifferential of a convex function and its conjugate", *Journal of Convex Analysis*, vol. 14, no. 3, pp. 479–514, 2007.

[5] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*. New York: Springer, 1998.

[6] J. Collier, "The dual of a space with the Radon-Nikodým property", *Pacific Journal of Mathematics*, vol. 64, no. 1, pp. 103–106, 1976.

[7] G. Colombo and L. Thibault, "Prox-regular sets and applications", in *Handbook of nonconvex analysis and applications*, D. Gao and D. Motreanu, Eds., Somerville, Mass: International Press, 2010, pp. 99–182.

[8] M. D. Contreras and R. Payá, "On upper semicontinuity of duality mappings", *Proceedings of the American Mathematical Society*, vol. 121, no. 2, pp. 451–459, 1994.

[9] R. Correa, Y. Garcia, and A. Hantoute, "Integration formulas via the Fenchel subdifferential of nonconvex functions", *Nonlinear Analysis*, vol. 75, no. 3, pp. 1188–1201, 2012.

[10] R. Correa, A. Hantoute, and D. Salas, "Integration of nonconvex epi-pointed functions in locally convex spaces", *Journal of Convex Analysis*, vol. 23, pp. 511–530, 2016.

[11] R. Correa and A. Jofré, "Some properties of semismooth and regular functions in nonsmooth analysis", in *Recent Advances in System Modelling and Optimization*, ser. Lecture Notes in Control and Information Sciences, L. Contesse, R. Correa, and A. Weintraub, Eds., vol. 87, Springer Berlin Heidelberg, 1986, pp. 69–85.

[12] R. Correa, D. Salas, and L. Thibault, "Smoothness of the metric projection onto nonconvex bodies in Hilbert spaces", *Journal of Mathematical Analysis and Applications*, 2016, (to appear).

[13] J. Diestel and J. J. Uhl, *Vector measures.* Providence, R.I: American Mathematical Society, 1977.

[14] S. Fitzpatrick and R. R. Phelps, "Differentiability of the metric projection in Hilbert space", *Transactions of the American Mathematical Society*, vol. 170, no. 2, pp. 483–501, 1982.

[15] G. Godefroy and V. Indumathi, "Strong proximinality and polyhedral spaces", *Revista Matemática Coplutense*, vol. XIV, no. 1, pp. 105–125, 2001.

[16] ——, "Norm-to-Weak upper semi-continuity of the duality and pre-duality mappings", *Set-Valued Analysis*, vol. 10, pp. 317–330, 2002.

[17] G. Godefroy, V. Montesinos, and V. Zizler, "Strong subdifferentiability of norms and geometry of Banach spaces", *Commentationes Mathematicae Universitatis Carolinae*, vol. 36, no. 6, pp. 493–502, 1995.

[18] J. B. Hiriart-Urruty and J. Benoist, "What is the subdifferential of the closed convex hull of a function?", *SIAM Journal on Mathematical Analysis*, vol. 27, no. 6, pp. 1661–1679, 1996.

[19] R. B. Holmes, "Smoothness of certain metric projections on Hilbert space", *Transactions of the American Mathematical Society*, vol. 184, pp. 87–100, 1973.

[20] J. B. Kruskal, "Two convex counterexamples: a discontinuous envelope function and a nondifferentiable nearest-point mapping", *Proceedings of the American Mathematical Society*, vol. 23, pp. 697–703, 1969.

[21] J. E. Martinez-Legaz, "Integration of the Fenchel subdifferentials revisited", *Vietnam Journal of Mathematics*, vol. 42, no. 4, pp. 533–542, 2014.

[22] M. Mazade, "Ensembles localement prox-réguliers et inéquations variationnelles", PhD thesis, Université Montpellier II, Montpellier, France, 2011.

[23] J. J. Moreau, "Proximité et dualité dans un espace hilbertien", *Bulletin de la Société Mathématique de France*, vol. 93, pp. 273–299, 1965.

[24] ——, "Fonctionnelles convexes", *Séminaire Jean Leray, Collège de France*, no. 2, pp. 1–108, 1966-1967.

[25] R. R. Phelps, *Convex functions, monotone operators, and differentiability*. Berlin New York: Springer-Verlag, 1989.

[26] R. A. Poliquin, R. T. Rockafellar, and L. Thibault, "Local differentiability of distance functions", *Transactions of the American Mathematical Society*, vol. 352, no. 11, pp. 5231–5249, 2000.

[27] J.-B. Poly and G. Raby, "Fonction distance et sigularités", *Bulletin des Sciences Mathématiques (2me Série)*, vol. 108, no. 2, pp. 187–195, 1984.

[28] R. T. Rockafellar, "Characterization of the subdifferentials of convex functions", *Pacific Journal of Mathematics*, vol. 17, pp. 497–510, 1966.

[29] D. Salas, "Convex smooth-like properties and Faces Radon-Nikodým property in Banach spaces", (Submitted).

[30] D. Salas and L. Thibault, "Characterizations of nonconvex sets with smooth boundary in terms of the metric projection in Hilbert spaces", (work in progress).

[31] H. Schaefer and M. Wolff, *Topological vector spaces*. New York: Springer, 1999.

[32] C. Stegall, "The duality between Asplund spaces and spaces with the Radon-Nikodým property", *Israel Journal of Mathematics*, vol. 29, no. 4, pp. 408–412, 1978.

[33] L. Thibault and D. Zagrodny, "Integration of subdifferentials of lower semicontinuous functions on Banach spaces", *Journal of Mathematical Analysis and Applications*, vol. 189, no. 1, pp. 33–58, 1995.

[34] ——, "Subdifferential determination of essentially directionally smooth functions in Banach space", *SIAM Journal on Optimization*, vol. 20, no. 5, pp. 2300–2326, 2010.

[35] L. Thibault and N. Zlateva, "Integrability of subdifferentials of directionally Lipschitz functions", *Proceedings of the American Mathematical Society*, vol. 133, no. 10, pp. 2939–2948, 2005.

# Part I

# Convex Geometry: Integration Formulas Using the Fenchel-subdifferential and the Faces Radon-Nikodým Property

# Introduction of Part I

The problem with integration of lower semicontinuous convex functions in terms of their Fenchel subdifferential, was started and solved by Moreau [33] in Hilbert spaces. The problem was then studied by Rockafellar [38] in the 60's: He proved that every two proper lower semicontinuous convex functions $f, g$ defined over a Banach space $X$ with values in $\mathbb{R} \cup \{+\infty\}$ satisfying

$$\partial f(x) \subseteq \partial g(x), \ \forall x \in X,$$

are in fact equal up to an additive constant. Motivated by this result, some generalizations to nonconvex cases have been made by, among others, Benoist, Burachik, Correa, Daniilidis, Jofré, Martinez-Legaz, Poliquin, Rocco, Thibault, Zagrodny and Zlateva involving the Fenchel subdifferential and other kinds of subdifferentials as well (see, for example [3], [12], [19], [31], [46]–[48]). The literature on the subject is vast. In particular, in the article of R. Correa, Y. Garcia and A. Hantoute [17], they accomplished many integration results for a particular family of nonconvex functions introduced by Benoist and Hiriart-Urruty in [30]: the ones which are *epi-pointed*. A function $f$ defined over a Banach space $X$ with values in $\mathbb{R} \cup \{+\infty\}$ is epi-pointed if the effective domain of its Legendre-Fenchel conjugate $f^*$ has nonempty interior.

The most interesting result in this line is the following: If $X$ is a Banach space with the Radon-Nikodým property (RNP, for short), then for each epi-pointed and lower-semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ and any function $g : X \to \mathbb{R} \cup \{+\infty\}$ we have that

$$\partial f(x) \subseteq \partial g(x), \ \forall x \in X \Rightarrow \exists c \in \mathbb{R}, \ \overline{\mathrm{co}}\, f = \overline{\overline{\mathrm{co}}\, g \square \sigma_{\mathrm{dom} f^*}} + c,$$

where $\overline{\mathrm{co}}\, f$ denotes the closed convex envelope of $f$, $\sigma_{\mathrm{dom} f^*}$ is the support function of the effective domain of $f^*$, $\mathrm{dom} f^*$, and $\square$ is the classic Moreau inf-convolution. The main observation that allowed the authors to derive this integration formula is the following: whenever $X$ is a Banach space having the RNP and $f$ is an epi-pointed lower semicontinuous function, then the set

$$D_1 := \{x^* \in \mathrm{int}(\mathrm{dom}\, f^*) \ : \ \partial f^*(x^*) = (\partial f)^{-1}(x^*)\}$$

is dense in $\mathrm{int}(\mathrm{dom}\, f^*)$, where the subdifferential of $f^*$ is defined as a set-valued mapping from $X^*$ into $X^{**}$, which is usually written as $\partial f^* : X^* \rightrightarrows X^{**}$.

The Radon-Nikodým property was introduced as first as a vector-valued measure property: A Banach space $X$ is said to have the RNP if for every measure space $(\Omega, \Sigma, \mu)$ with $\mu$ being a countable-additive measure over the $\sigma$-algebra $\Sigma$, and every vector measure $m : \Sigma \to X$ which is absolutely $\mu$-continuous, there exists a Bochner $\mu$-integrable function $f : \Omega \to X$ such that

$$m(E) = \int_E f(\omega) d\mu(\omega), \qquad \forall E \in \Sigma.$$

This subject is deeply developed in the books [9] and [21]. Nowadays, Radon-Nikodým property is likely to be presented according to some of its many geometrical characterizations. Here, we present it in terms of convex closed hulls of *strongly-exposed points* (see Definitions 2.2.9 and 2.2.10 in Chapter 2): We will say that a Banach space $X$ has the RNP if and only if every closed convex bounded set $K$ of $X$ coincides with the closed convex hull of its strongly-exposed points.

In Chapter 1 we generalize the result of Correa-García-Hantoute, replacing the Banach space by a general (Hausdorff) locally convex space $(X, \theta)$. For this we introduce a new notion of epi-pointedness, depending on the topology used in the dual space $X^*$, and replace the RNP hypothesis with a special property of the function itself, namely to be *Subdifferential Dense Primal Determined* (SDPD, for short). The main idea of these functions is to replace the condition $\partial f^*(x^*) = (\partial f)^{-1}(x^*)$ by

$$\partial f^*(x^*) = \overline{\text{co}}^{w^{**}} \left[ (\partial f)^{-1}(x^*) \right],$$

which, under suitable notions of density and epi-pointedness, will be enough to ensure the desired integration formula (for more details, see 1.3.7 in Chapter 1). The obtained integration formula is a generalization of Correa-Garcia-Hantoute integration theorem. Indeed, as we observe in Proposition 1.3.13, in a Banach space with the RNP, every epi-pointed lower semicontinuous function is SDPD. Nevertheless, outside the RNP setting, lower semicontinuity is not enough to guarantee SDPD (see Proposition 1.3.6).

A natural question arises: How can we assure that SDPD condition is satisfied by a large family of epi-pointed functions? This question is what leads Chapters 2 and 3. We realize that a minimal expected condition over the space is that SDPD condition be satisfied in the convex setting, and so we introduce the notion of SDPD spaces: A Banach space $X$ is said to be SDPD if and only if every epi-pointed convex lower-semicontinuous function defined on $X$ is an SDPD function.

SDPD spaces are very similar to Banach spaces with a $w^*$-Asplund dual space. Recall that a Banach space $X$ (resp. a dual space $X^*$) is said to be an Asplund space (resp. $w^*$-Asplund space) if for every real-valued convex continuous (resp. $w^*$-lower semicontinuous) function $f$ over $X$ (resp. $X^*$) is Fréchet-differentiable in a dense set of $X$ (resp. $X^*$). These spaces are in duality with the RNP, namely:

1. A Banach space $X$ is an Asplund space if and only if its dual $X^*$ has the RNP. (see Theorem 2.2.13).

2. A Banach space $X$ has the RNP if and only if its dual $X^*$ is a $w^*$-Asplund space (see Theorem 2.2.15).

Even though the definitions of the RNP and Asplund spaces consider a wide family of sets or functions, many reductions have been made. In fact, it is known that $X^*$ is a $w^*$-Asplund space if and only if every equivalent dual norm on $X^*$ is Fréchet-differentiable at some point. Analogously, $X$ is an Asplund space if and only if the same differentiability condition holds for every equivalent norm on $X$.

Chapter 2 is devoted to generalize these kinds of theorems to more general properties of Banach spaces. Assume that we have a property $(\mathcal{P})$ over the class of Banach spaces of the following form:

A Banach space $X$ has the property $(\mathcal{P})$ if for every real-valued convex continuous function $f$ over $X$, there exists a dense set $D$ such that $f$ satisfies certain condition $\mathcal{P}_X(f,x)$ at every $x \in D$.

In the case of Asplund property, the condition $\mathcal{P}_X(f,x)$ is $f$ to be Fréchet-differentiable at $x$. We will show that, under certain regularity conditions of $(\mathcal{P})$, the reductions known for Asplund property are still valid, that means, we will obtain theorems of the form "A Banach space $X$ has the property $(\mathcal{P})$ if and only if every equivalent norm $p$ over $X$ satisfies $\mathcal{P}_X(p,x)$ at every point $x$ in a dense set $D$".

The Banach space properties that have this regularity condition (which will be formally defined later) are called *convex smooth-like properties*. As for Asplund property, we will also define their weak-star version in the dual space, called *convex $w^*$-smooth-like properties*. Finally, we will show that both, convex smooth-like and convex $w^*$-smooth-like properties, have a dual (or predual) *geometrical interpretation*, namely, that they are in duality with some property of the convex $w^*$-closed (resp. closed) sets of the dual space (resp. predual space).

In Chapter 3, we show that SDPD property is in fact a convex $w^*$-smooth-like property and we provide its geometrical interpretation: what we call the *Faces Radon-Nikodým property (FRNP)*.

We also introduce a stronger version of the FRNP and we connect it with the notion of *strong subdifferentiability*, which has been widely studied in the literature (see, e.g., [27], [29] and the references therein). At the end of Chapter 3, we study the natural question on this theory: Whether or not the FRNP and the RNP are equivalent. We provide several partial results in this line but the question remains open.

# Chapter 1

# Integration Formulas for Epi-pointed Functions in Locally Convex Spaces

In what follows, we will write $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$. Also, $\tau_0$ will denote the usual topology in $\mathbb{R}$ (and its natural extension to $\mathbb{R}_\infty$ and $\overline{\mathbb{R}}$). For a topological space $(T, \tau)$ and a point $t_0 \in T$, we will denote by $\mathcal{N}_T(t_0, \tau)$ (or simply by $\mathcal{N}_T(t_0)$, $\mathcal{N}(t_0, \tau)$ or $\mathcal{N}(t_0)$ if there is no confusion) the set of all neighborhoods of $t$ in $(T, \tau)$. Sometimes it will be useful to say that an element $V \in \mathcal{N}_T(t_0, \tau)$ is a *$\tau$-neighborhood of* $t_0$.

We will say that a subset $\mathfrak{B} \subset \mathcal{N}_T(t_0, \tau)$ is a *fundamental system* (or neighborhood basis) of $\mathcal{N}_T(t_0, \tau)$ if

$$\forall U \in \mathcal{N}_T(t_0, \tau),\ \exists V \in \mathfrak{B} \text{ such that } V \subseteq U.$$

If $(S, \theta)$ is another topological space, we will denote by $\tau \times \theta$ the product topology over $T \times S$. Also, if $T_0$ is a nonempty set of $T$, we will denote by $\tau\big|_S$ the induced topology on $S$ by $\tau$. Sometimes, if there is no confusion, we will write $(T_0, \tau)$ instead of $\left(T_0, \tau\big|_{T_0}\right)$ to denote the topological subspace of $(T, \tau)$ given by $T_0$.

For a subset $A \subseteq T$, we denote by $\operatorname{int}_\tau A$, $\operatorname{cl}_\tau A$ and $\operatorname{bd}_\tau A$ (or simply $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{bd} A$, if there is no confusion), the interior, closure and boundary of $A$, respectively. Sometimes, we will write $\overline{A}^\tau$ (or simply $\overline{A}$) instead $\operatorname{cl}_\tau A$ to denote the closure of $A$, and write $\operatorname{int}(A, \tau)$ instead $\operatorname{int}_\tau(A)$ to denote te interior of $A$.

For a metric space $(M, d)$ we denote by $\tau_d$ the topology induced by $d$ on $M$. Analogously, for a normed vector space $(E, \|\cdot\|)$ we denote by $\tau_{\|\cdot\|}$ the topology induced by $\|\cdot\|$ on $E$.

For a subset $S$ of a topological vector space $(X, \theta)$, we will write $\operatorname{co} S$ and $\overline{\operatorname{co}}^\theta(S)$ to denote the *convex hull* and *$\theta$-closed convex hull* of $S$, respectively. If there is no confusion, we may omit the topology and simply write $\overline{\operatorname{co}}\, S$ to denote the closed convex hull of $S$.

## 1.1 Locally Convex Spaces

In this section we briefly summarize the fundamental notions and topology properties on *Real Locally Convex Spaces*. In what follows, $\theta$ will be a locally convex topology over the real vector space $X$, and whenever we say a topology is locally convex, we will assume that it is also a Hausdorff topology.

Let $X$ be a real vector space and let us denote by $X'$ its *algebraic dual space*, that is, the real vector space of all linear functionals defined over $X$. In this section we will study the subset of all $\theta$-continuous linear functionals over $X$ when we endow $X$ with a locally convex topology $\theta$. These objects are one of the foundations of convex analysis, mainly provided by their strong applications via the celebrated Hahn-Banach theorems.

**Lemma 1.1.1** *Let $(X, \theta)$ be a locally convex space and let $f : X \to \mathbb{R}$ be a sublinear functional. The following assertions are equivalent:*

*(a) $f$ is $\theta$-$\tau_0$-continuous.*

*(b) $f$ is $\theta$-$\tau_0$-continuous at $0$.*

*(c) $f$ is uniformly $\theta$-$\tau_0$-continuous.*

*(d) $f^{-1}\big(]-1,1[\big)$ is a $\theta$-open neighborhood of $0$.*

*(e) There exists a $\theta$-$\tau_0$-continuous seminorm $\rho$ such that $|f| \leq \rho$.*

*Furthermore, if $f$ is linear, then it is $\theta$-$\tau_0$-continuous if and only if its kernel $\mathrm{Ker}(f)$ is a $\theta$-closed subspace of $X$.*

In what follows, we will write $(X, \theta)^*$ (or simply $X^*$) the *topological dual space* of $(X, \theta)$, that is, the subset of $X'$ given by all $\theta$-continuous linear functionals over $X$. Also, we will denote by $w(X, X^*)$ (or simply $w$), and $w^*(X^*, X)$ (or simply $w^*$) the weak topology on $X$ induced by $X^*$ and the weak-star topology on $X^*$ induced by $X$, respectively.

It is known that for a locally convex space $(X, \theta)$, $w(X, X^*) \subseteq \theta$ and that

1. $(X, w(X, X^*))^* = X^*$ and $w(X, X^*)$ is the coarsest locally convex topology on $X$ enjoying this property.

2. $(X^*, w^*(X^*, X))^* = X$ and $w^*(X^*, X)$ is the coarsest locally convex topology on $X^*$ enjoying this property.

Depending on which locally convex topology we endow $X$ with, the dual space $X^*$ may change. To specify which dual we are working with, we need the notion of *dualities*. For a subspace $Y$ of the algebraic dual $X'$, we say that $X$ and $Y$ are in duality if there exists a locally convex topology $\theta(Y)$ such that $(X, \theta(Y))^* = Y$. In such a case, we will write $\langle X, Y\rangle$ to denote this duality (with duality product $\langle\cdot,\cdot\rangle : Y \times X \to \mathbb{R}$ given by $\langle y, x\rangle = y(x)$).

Note that this notion is not topological: Indeed, $X$ and $Y$ are in duality if and only if the weak topology $w(X, Y)$ is Hausdorff, that is, if and only if

$$\forall x \in X \setminus \{0\},\ \exists y \in Y \setminus \{0\} \text{ such that } \langle y, x\rangle \neq 0. \tag{1.1}$$

Thus, the notation $\langle X, Y\rangle$ is not ambiguous. This objects are called dualities because of the dual relations given by the weak and weak-star topologies. Observe also that $X$ and $Y$ can interchange the roles of primal and dual space, indistinctly. Thus, we won't make a difference between de duality $\langle X, Y\rangle$ and the duality $\langle Y, X\rangle$.

**Proposition 1.1.2** *Let $(X, \theta)$ be a locally convex space and let $C$ be a $\theta$-closed convex subset of $X$. Then, $C$ is $w$-closed.*

*In particular, for every subset $S$ of $X$, we have that*

$$\overline{\text{co}}^{\theta}(S) = \overline{\text{co}}^{w}(S).$$

By the latter proposition, if $(X, \theta)$ is a locally convex space and $S \subseteq X$, we can write $\overline{\text{co}}\, S$ to denote the closed convex hull of $S$, regardless which topology we are using (in between $w(X, X^*)$ and $\theta$).

Another important feature of convex sets in locally convex spaces is that, whenever they are *compact*, they can be constructed only from their *extreme points*: These are the Krein-Milman Theorem and Milman Theorem, that we present below. Due to the complexity of their proves and also since both are well-known theorems, we will limit ourself only to present them, referring the reader to [23, Ch. 3] for further information.

**Definition 1.1.3** (Extreme point) *Let $X$ be a real vector space. For a subset $S$ (not necessarily convex) of $X$, a point $\bar{x} \in S$ is said to be an* extreme point *of $S$ if*

$$\forall x_1, x_2 \in S,\ \bar{x} \in [x_1, x_2] \Rightarrow x_1 = x_2 = \bar{x}.$$

*where $[x_1, x_2] := \{tx_1 + (1-t)x_2 \ :\ t \in [0, 1]\}$. We denote by $\text{ext}(S)$ the set of extreme points of $S$.*

Note that the notion of extreme points is completely algebraic but it has topological implications, just like convexity.

**Theorem 1.1.4** (Krein-Milman) *Let $(X, \theta)$ be a locally convex space and $K$ be a convex $\theta$-compact subset of $X$. We have that* *(a)* $\mathrm{ext}(K) \neq \emptyset$.

*(b)* $K = \overline{\mathrm{co}}(\mathrm{ext}(K))$.

*Proof.* See [23, Theorem 3.65]. □

**Theorem 1.1.5** (Milman) *Let $(X, \theta)$ be a locally convex space and $K$ be a convex $\theta$-compact subset of $X$. Then, for every subset $S \subseteq K$ such that $K = \overline{\mathrm{co}}(S)$, we have that*

$$\mathrm{ext}(K) \subseteq \mathrm{cl}\, S.$$

*In particular,* $\mathrm{ext}(K) \subseteq \mathrm{ext}(\mathrm{cl}\, S)$.

*Proof.* See [23, Theorem 3.66]. □

**Definition 1.1.6** *Let $(X, \theta)$ be a locally convex space and let $S$ be a subset of $X$. We define the* polar set of $S$ with respect to the duality $\langle X, X^* \rangle$ *as the set*

$$S^o := \{x^* \in X^* \ : \ \langle x^*, x \rangle \leq 1, \ \forall x \in S\}.$$

*We also define the* bipolar set of $S$ with respect to the duality $\langle X, X^* \rangle$, *as the polar set of $S^o$ with respect to the duality $\langle X, X^* \rangle$, that is,*

$$S^{oo} := \{x \in X \ : \ \langle x^*, x \rangle \leq 1, \ \forall x^* \in S^o\}.$$

If we have three vector spaces $X, Y$ and $Z$, and two dualities $\langle X, Y \rangle$ and $\langle X, Z \rangle$, the notation of the polar set $S^o$, for some set $S \subseteq X$, is ambiguous. In such a case, we will specify respect to which duality we are taking the polar set.

**Theorem 1.1.7** (Bipolar theorem) *Let $(X, \theta)$ be a locally convex space, $X^*$ be its dual space and $S$ be a subset of $X$. We have that*

$$S^{oo} = \overline{\mathrm{co}}(S \cup \{0\}).$$

*Analogously, if $S \subseteq X^*$, then $S^{oo} = \overline{\mathrm{co}}^{w^*}(S \cup \{0\})$, where the bipolar set of $S$ is taken with respect to the duality $\langle X, X^* \rangle$.*

**Theorem 1.1.8** (Alaoglu-Bourbaki) *Let $(X, \theta)$ be a locally convex space, $X^*$ be its dual space. For any $V \in \mathcal{N}_X(0, \theta)$, we have that $V^o$ is $w^*$-compact.*

*Proof.* See [23, Theorem 3.37] □

We will now introduce some other natural locally convex topologies that can be constructed from the duality $\langle X, X^* \rangle$.

**Definition 1.1.9** *Let $(X,\theta)$ be a locally convex space. We define*

(a) *The* Mackey topology on $X$ induced by $X^*$, *denoted by $\tau(X,X^*)$, as the locally convex topology induced by the family of seminorms $\{p_K \ : \ K\, w^*\text{-compact subset of } X^*\}$, where, for each $K$ $w^*$-compact subset of $X^*$, $p_K : X \to \mathbb{R}_+$ is given by*

$$p_K(x) := \sup\{|\langle x^*, x\rangle| \ : \ x^* \in K\}.$$

(b) *The* Mackey topology on $X^*$ induced by $X$, *denoted by $\tau(X^*,X)$, as the locally convex topology induced by the family of seminorms $\{p_K \ : \ K\, w\text{-compact subset of } X\}$, where, for each $K$ $w$-compact subset of $X$, $p_K : X^* \to \mathbb{R}_+$ is given by*

$$p_K(x^*) := \sup\{|\langle x^*, x\rangle| \ : \ x \in K\}.$$

**Remark 1.1.10** Let us note that the Mackey topology $\tau(X,X^*)$ is a locally convex topology on $X$: Indeed, $\tau(X,X^*)$ is induced by seminorms and, since every singleton in $X^*$ is $w^*$-compact, it is also Hausdorff.

Observe also that the associated convergence to the topology $\tau(X,X^*)$ corresponds to the *uniform convergence on $w^*$-compact sets*, that is, a net $(x_\alpha)_{\alpha\in\Lambda} \subseteq X$ $\tau(X,X^*)$-converges to a point $x \in X$ if and only if

$$\sup_{x^*\in K} \langle x^*, x_\alpha - x\rangle \to 0, \qquad \forall K\, w^*\text{-compact set of } X^*.$$

The same remarks can be done for the Mackey topology $\tau(X^*,X)$ on $X^*$, replacing $(X,\theta)$ and $(X^*,w^*)$ by $(X^*,w^*)$ and $(X,w)$, respectively.

**Proposition 1.1.11** *Let $(X,\theta)$ be a locally vector space and let $X^*$ be its dual space. The followings hold:*

(a) *$(X,\tau(X,X^*))^* = X^*$, and $\tau(X,X^*)$ is the finest locally convex topology on $X$ enjoying this property.*

(b) *$(X^*,\tau(X^*,X))^* = X$, and $\tau(X^*,X)$ is the finest locally convex topology on $X^*$ enjoying this property.*

**Proposition 1.1.12** *Let $(X,\theta)$ be a locally convex space, let $X^*$ be its dual space and let $S$ be a subset of $X$. Then, $S$ is $\theta$-bounded if and only if $S$ is $w(X,X^*)$-bounded.*

*In particular, the family of $w(X,X^*)$-bounded sets and the family of $\tau(X,X^*)$-bounded sets coincide.*

*Proof.* By definition of bounded sets it is not hard to see that for two locally convex topologies $\theta_1$ and $\theta_2$ on $X$ satisfying $\theta_1 \subseteq \theta_2$ (that is, $\theta_2$ is finer than $\theta_1$), we have that every $\theta_2$-bounded set is also $\theta_1$-bounded. Thus, to prove this proposition, it is enough to show that if $S$ is $w(X, X^*)$-bounded, then it is $\tau(X, X^*)$-bounded.

Assume that $S$ is $w(X, X^*)$-bounded and fix $V \in \mathcal{N}_X(0, \tau(X, X^*))$. Without lose of generality, we may assume that $V$ is closed and absolutely convex, and so $V = K^o$ for some $K$ $w^*$-compact, absolutely convex subset of $X^*$. Note that $V = p^{-1}([0, 1])$, where $p : X \to \mathbb{R}_+$ is the seminorm given by

$$p(x) = \sup_{x^* \in K} |\langle x^*, x\rangle| = \sup_{x^* \in K} \langle x^*, x\rangle,$$

where the second equality follows from the symmetry of $K$. Let us denote by $X_0$ the quotient space $X/\operatorname{Ker}(p)$ and let $\pi_0 : X \to X_0$ be the quotient map. Define the functional $\|\cdot\|_0 : X_0 \to \mathbb{R}_+$ given by $\|[x]\|_0 = p(x)$ (where $[x]$ denotes the equivalent class of $x$, for all $x \in X$). It is not hard to see that $\|\cdot\|_0$ is a norm on $X_0$.

Let us endow $X_0$ with the topology $\tau_{\|\cdot\|_0}$ and denote $X_0^*$ as the associated dual space. We will prove that $\pi_0(S)$ is bounded in $(X_0, \|\cdot\|_0)$. First note that for every $\phi \in X_0^*$, the functional $\phi \circ \pi_0 \in X^*$: Indeed, it is enough to note that $|\phi \circ \pi_0| \leq \|\phi\|_{0*} p$, where $\|\cdot\|_{0*}$ is the dual norm on $X_0^*$ associated to $\|\cdot\|_0$. Then, since $S$ is $w(X, X^*)$-bounded, there exists $\lambda > 0$ such that $S \subseteq \lambda U_\phi$, where $U_\phi = \{x \in X \ : \ |\phi \circ \pi_0(x)| < 1\}$. Then,

$$\pi_0(S) \subseteq \lambda \pi_0(U_\phi) = \{[x] \in X_0 \ : \ |\phi([x])| < 1\}.$$

This entails, by the arbitrariness of $\phi$, that $\pi_0(S)$ is $w(X_0, X_0^*)$-bounded. Then, applying the Principle of Uniform Boundedness (see, e.g., [16, Ch. 3 - §14]) to $\pi_0(S)$ (as a subset of $X_0^{**}$), we conclude that there exists $M > 0$ such that

$$\sup\{\|[x]\|_0 \ : \ [x] \in \pi_0(S)\} \leq M.$$

This yields that for every $x \in S$, $p(x) = \|[x]\|_0 \leq M$ and so, $S \subseteq MV$. This proves that $S$ is $\tau(X, X^*)$-bounded, as we wanted to show. □

**Definition 1.1.13** *Let $(X, \theta)$ be a locally convex space. We define the* Strong topology on $X^*$ induced by $X$, *denoted by $\beta(X^*, X)$ (or simply $\beta$, if there is no confusion), as the locally convex topology induced by the family of seminorms $\{p_B \ : \ B \ \theta\text{-bounded set of } X\}$, where, for each $B$ $\theta$-bounded subset of $X$, $p_B : X^* \to \mathbb{R}$ is given by*

$$p_B(x^*) = \sup_{x \in B} |\langle x^*, x\rangle|.$$

*We define the* bidual space of $(X, \theta)$, $(X, \theta)^{**}$ *(or simply $X^{**}$, if there is no confusion), as the dual space $(X^*, \beta(X, X^*))^*$.*

Note that, since the bounded sets for the topologies $w(X, X^*)$, $\theta$ and $\tau(X, X^*)$ coincide, the definition of the strong topology $\beta(X^*, X)$ and the bidual space $X^{**}$ is not ambiguous.

Also, in general, $\tau(X^*, X) \subsetneq \beta(X^*, X)$: Indeed, it is enough to consider a non-reflexive Banach space $(X, \|\cdot\|)$ and note that the topology $\beta(X^*, X)$ coincides with the topology $\tau_{\|\cdot\|_*}$, where $\|\cdot\|_*$ is the dual norm on $X^*$.

Finally, note that we can endow $X^*$ with new topologies induced by the bidual space $X^{**}$: Namely, the weak topology $w(X^*, X^{**})$ and the Mackey topology $\tau(X^*, X^{**})$. We have then the following order of topologies on the dual space $X^*$:

$$\underbrace{w^*(X^*, X) \subseteq \tau(X^*, X)}_{\text{duality } \langle X, X^* \rangle} \qquad \text{and} \qquad \underbrace{w(X^*, X^{**}) \subseteq \beta(X^*, X) \subseteq \tau(X^*, X^{**})}_{\text{duality } \langle X^*, X^{**} \rangle}. \tag{1.2}$$

We can iterate this process to construct the third dual $X^{3*}$, the forth dual $X^{4*}$, and so on. Nevertheless, we won't need to do so. We will work only until the bidual $X^{**}$, endowed with the weak-star topology $w^*(X^*, X^{**})$ or the Mackey topology $\tau(X^*, X^{**})$. We will denote the weak-star topology on $X^{**}$ by $w^{**}$, in order to remark that it corresponds to a bidual topology.

## 1.2 Some Fundamentals of Convex Analysis

In this section we present the fundaments convex analysis, that is, the fundamental properties of convex functionals defined over a locally convex space $(X, \theta)$. This presentation is based, historically speaking, on the fundamental contributions of J. J. Moreau which are contained in [34]. We also refer the reader to the books [5], [36], [39] and [51], for other approaches. Here, we adopt the conventions and $(+\infty) \cdot 0 = (-\infty) \cdot 0 = 0$. Recall that we say that the sum $a + b$ is well defined for two elements $a, b \in \overline{\mathbb{R}}$ if it is not of the form $(+\infty) + (-\infty)$ or $(-\infty) + (+\infty)$. Also, we will use the conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

In what follows, for a topological space $(T, \tau)$ a function $f : T \to \overline{\mathbb{R}}$ we will denote by $\overline{f}^{\tau}$ or by $\mathrm{cl}_\tau f$ the *closure of* $f$, that is,

$$\mathrm{cl}_\tau f := \sup \left\{ g : T \to \overline{\mathbb{R}} \;:\; g \ \tau\text{-lower semicontinuous}, \ g \leq f \right\}.$$

If there is no confusion, we will write simply $\overline{f}$ or $\mathrm{cl}\, f$. Recall that a function $f : T \to \overline{\mathbb{R}}$ is $\tau$-lower semicontinuous ($\tau$-lsc, for short), if its epigraph $\operatorname{epi} f$ is $(\tau \times \tau_0)$-closed in $T \times \mathbb{R}$. Since the supremum of $\tau$-lsc functions is also $\tau$-lsc, we have that $\mathrm{cl}_\tau f$ is the largest $\tau$-lsc function satisfying $\mathrm{cl}_\tau f \leq f$.

Also, for a function $f : T \to \overline{\mathbb{R}}$ we denote its *(effective) domain* and its *epigraph* as the sets

$$\operatorname{dom} f := \{t \in T \;:\; f(t) < +\infty\} \quad \text{and} \quad \operatorname{epi} f := \{(t, r) \in T \times \mathbb{R} \;:\; f(t) \leq r\}.$$

### 1.2.1 Convex lower semicontinuous functions

**Definition 1.2.1** *Let $f : X \to \overline{\mathbb{R}}$ be a extended real-valued function. We say that $f$ is* convex *if $\operatorname{epi} f$ is a convex subset of $X \times \mathbb{R}$.*

Whenever $f : X \to \overline{\mathbb{R}}$ is proper, it is not hard to verify that $f$ is convex if and only if

$$\forall x, y \in X, \forall t \in [0,1],\ f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \tag{1.3}$$

which is the classic definition of convexity for real-valued functions.

**Example 1.2.2** (Convex Functions) We present some examples of convex functions that will be useful in what follows:

(a) Every sublinear function is convex. In particular, seminorms and Minkowski functionals are convex, where the Minkowski functional associated to an absorbing convex set $C$ of a vector space $X$ is given by

$$\begin{aligned} \rho_C : X &\to [0, +\infty[ \\ x &\mapsto \rho_C(x) := \inf\{\lambda > 0 \ : \ x \in \lambda C\}. \end{aligned}$$

(b) *Indicator function*: For a subset $S \subseteq X$ we define the indicator function of $S$ as the function $I_S : X \to \mathbb{R}_\infty$ given by

$$I_S(x) := \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{if } x \notin S. \end{cases}$$

We have that $S$ is a convex set if and only if $I_S$ is a convex function.

(c) Let $\{f_i : \ i \in I\}$ be a family of extended real-valued functions over $X$. If all functions $f_i$ are convex, then $\sup_{i\in I} f_i$ is also convex.

(d) *Inf-Convolution:* Let $f, g : X \to \mathbb{R}_\infty$ be two proper functions. We define the (Moreau) inf-convolution of $f$ and $g$ as the extended real-valued function

$$\begin{aligned} f\Box g : X &\to \overline{\mathbb{R}} \\ x &\mapsto (f\Box g)(x) := \inf\{f(y) + g(x-y) \ : \ y \in X\}. \end{aligned}$$

Clearly, $f\Box g = g\Box f$. Also, if $f$ and $g$ are both convex, then $f\Box g$ is convex.

(e) *Support functional*: Let $(X, \theta)$ be a locally convex space with dual space $X^*$, and let $K$ be a subset of $X^*$. We define the support functional of $K$ as the extended real-valued function $\sigma_K : X \to \overline{\mathbb{R}}$ given by

$$\sigma_K(x) := \sup\{\langle x^*, x\rangle \ : \ x^* \in K\}.$$

Regardless the structure of $K$, $\sigma_K$ is always sublinear (and therefore, convex). Furthermore,

$$\sigma_K = \sigma_{\mathrm{cl}_{w^*}(K)} = \sigma_{\overline{\mathrm{co}}^{w^*}(K)}.$$

Note that support functionals also can be defined over $X^*$ for subsets of $X$ (duality $\langle X, X^*\rangle$), over $X^{**}$ for subsets of $X^*$, or over $X^*$ for subsets of $X^{**}$ (duality $\langle X^*, X^{**}\rangle$). We will always use the notation $\sigma_K$, and will specify over which space is defined, if necessary.

Observe that the seminorms $p_K$ used in Definition 1.1.9(a) are in fact support functions. Indeed, for $K \subseteq X^*$ $w^*$-compact, we have that

$$p_K(\cdot) := \sup\{|\langle x^*, \cdot\rangle| \ : \ x^* \in K\} = \sigma_{\overline{\mathrm{co}}^{w^*}(K\cup(-K))}.$$

Therefore, the Mackey topology $\tau(X, X^*)$ is the induced locally convex topology given by the family of support functionals $\sigma_K$, where $K$ is an absolutely convex $w^*$-compact subset of $X^*$.

Following the notation introduced in [34], we define the following families if functions:

$$\Gamma(X,\theta) := \big\{f : X \to \overline{\mathbb{R}} \ : \ f \text{convex and } \theta\text{-lsc}\big\} \tag{1.4}$$

$$\Gamma_0(X,\theta) := \big\{f : X \to \overline{\mathbb{R}} \ : \ f \text{convex, proper and } \theta\text{-lsc}\big\} \tag{1.5}$$

We will also denote by $\overline{\omega}$ and $\underline{\omega}$ the constant functions $\overline{\omega} \equiv \infty$ and $\underline{\omega} \equiv -\infty$. We will use this notation regardless the space over which they are defined.

**Definition 1.2.3** (Closed convex hull of functions) *Let $(X, \theta)$ be a locally convex space and let $f : X \to \overline{\mathbb{R}}$ be any extended real-valued function. We define the* $\theta$-closed convex hull of $f$ *as the function*

$$\overline{\mathrm{co}}^{\theta} f = \sup\{g : X \to \overline{\mathbb{R}} \ : \ g \in \Gamma(X,\theta),\ g \le f\}.$$

Clearly, by Example 1.2.2, it is clear that $\overline{\mathrm{co}}^{\theta} f \in \Gamma(X, \theta)$.

The following proposition characterizes the convex closed hull of a function in terms of the convex closed hull of its epigraph.

**Proposition 1.2.4** *Let $(X, \theta)$ be a locally convex space and let $f : X \to \overline{\mathbb{R}}$ be any extended real-valued function. We have that*

$$\overline{\mathrm{co}}^{\theta} f(x) = \inf\big\{r \in \mathbb{R} \ : \ (x,r) \in \overline{\mathrm{co}}^{\theta\times\tau_0}(\operatorname{epi} f)\big\}$$

*In particular, $\overline{\mathrm{co}}^{\theta} f$ is the (unique) function in $\Gamma(X, \theta)$ satisfying* $\operatorname{epi}\big(\overline{\mathrm{co}}^{\theta} f\big) = \overline{\mathrm{co}}^{\theta\times\tau_0}(\operatorname{epi} f)$.

**Remark 1.2.5** By Proposition 1.1.2, we know that the family of $(\theta \times \tau_0)$-closed convex sets of $X \times \mathbb{R}$ coincides with the family of $(w \times \tau_0)$-closed convex sets of $X \times \mathbb{R}$. Therefore, using Proposition 1.2.4, we have that for every extended real-valued function $f : X \to \overline{\mathbb{R}}$,

$$\overline{\mathrm{co}}^{w} f = \overline{\mathrm{co}}^{\theta} f = \overline{\mathrm{co}}^{\tau(X,X^*)} f.$$

This yields also that $\Gamma(X, w) = \Gamma(X, \theta) = \Gamma(X, \tau(X, X^*))$ and that $\Gamma_0(X, w) = \Gamma_0(X, \theta) = \Gamma_0(X, \tau(X, X^*))$. In what follows, for any locally convex topology $\theta'$ inducing the same duality $\langle X, X^*\rangle$, we will denote by $\overline{\mathrm{co}}\, f$ the $\theta'$-closed convex hull of $f$ and by $\Gamma(X)$ (resp.

$\Gamma_0(X)$) the set of all extended real-valued convex $\theta'$-lsc (resp. convex proper $\theta'$-lsc) functions over $X$.

Applying the latter observations in the case of dual spaces $X^*$ we introduce the following simpler notation: We will write $\Gamma(X^*)$ and $\Gamma_0(X^*)$ to denote the families $\Gamma(X^*, \beta(X^*, X^{**}))$ and $\Gamma_0(X^*, \beta(X^*, X^{**}))$, respectively. If $\theta'$ is a locally convex topology on $X^*$ with $w^* \subseteq \theta' \subseteq \tau(X^*, X)$, we will write $\Gamma(X^*, w^*)$ and $\Gamma_0(X^*, w^*)$ instead of $\Gamma(X^*, \theta')$ and $\Gamma_0(X^*, \theta')$.

**Remark 1.2.6** Noting that for every set $K$ of $(X, \theta)$ we can write $\overline{\text{co}}\, K = \overline{\text{co}\, K}^{\theta}$, we easily deduce from Proposition 1.2.4 that for every *convex function* $f : X \to \overline{\mathbb{R}}$, we have that

$$\overline{\text{co}}\, f = \overline{f}^{\theta}.$$

Thus, for convex functions $f : X \to \overline{\mathbb{R}}$ we can simply write $\overline{f}$ to denote its closure with respect to any locally convex topology in between $w(X, X^*)$ and $\tau(X, X^*)$.

**Proposition 1.2.7** *Let $f : X \to \overline{\mathbb{R}}$ be a proper convex function. Fix $x_0 \in \operatorname{dom} f$. The following assertions are equivalent:*

*(a) $f$ is $\theta$-$\tau_0$-continuous at $x_0$.*

*(b) $f$ is bounded above in a $\theta$-neighborhood of $x_0$.*

*(c) $f$ is* $\theta$-Lipschitz-continuous *near $x_0$, that is, there exists a $\theta$-continuous seminorm $\rho : X \to \mathbb{R}_+$, a neighborhood $U \in \mathcal{N}_X(x_0, \theta)$, and a constant $K > 0$ such that*

$$|f(x) - f(y)| \leq K\rho(x - y), \quad \forall x, y \in U. \tag{1.6}$$

*(d) $f$ is $\theta$-$\tau_0$-continuous at each point of* $\operatorname{int}(\operatorname{dom} f)$ *and* $x_0 \in \operatorname{int}(\operatorname{dom} f)$.

In what follows, for a proper function $f : X \to \mathbb{R}_\infty$ we will denote

$$\text{Cont}[f, \theta] := \{x \in \operatorname{dom} f \ : \ f \text{ is } \theta\text{-}\tau_0\text{-continuous at } x\}. \tag{1.7}$$

If there is no confusion, then we will simply write $\text{Cont}[f]$. The latter proposition shows that if $f$ is convex, then

$$\text{Cont}[f, \theta] = \operatorname{int}_\theta(\operatorname{dom} f).$$

Nevertheless, even if $f$ is convex, $\text{Cont}[f, \theta]$ can be empty even if $\operatorname{int}(\operatorname{dom} f)$ is not: Consider for example an infinite-dimensional Banach space $X$ and its dual space $X^*$ endowed with the $w^*$-topology. Then, the Minkowski functional $\rho_{\mathbb{B}_{X^*}} : X \to \mathbb{R}_+$ is a seminorm which is not $w^*$-continuous, since $\rho_{\mathbb{B}_X^*}^{-1}([0, 1]) = \mathbb{B}_{X^*}$, which is known to not to be a $w^*$-neighborhood of 0 (and then $\rho_{\mathbb{B}_{X^*}}$ cannot be $w^*$-continuous by Lemma 1.1.1). Noting that $\operatorname{int}(\operatorname{dom} \rho_{\mathbb{B}_{X^*}}) = X^*$, this proves our claim.

**Corollary 1.2.8** *Let $(X, \theta)$ be a locally convex space and $f : X \to \mathbb{R}_\infty$ be a proper convex function. Then,* $\operatorname{Cont}[f, \theta]$ *is nonempty if and only if* $\operatorname{int}_{(\theta \times \tau_0)}(\operatorname{epi} f)$ *is nonempty.*

**Definition 1.2.9** (Legendre-Fenchel Conjugate) *Let $(X, \theta)$ be a locally convex space and $f : X \to \overline{\mathbb{R}}$ be an extended real-valued function. We define the* (Legendre-Fenchel) conjugate of $f$ *as the extended real-valued function $f^* : X^* \to \overline{\mathbb{R}}$ given by*

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) \ : \ x \in X\} = \sup\{\langle x^*, x\rangle - f(x) \ : \ x \in \operatorname{dom} f\}.$$

*Analogously, we define the* biconjugate of $f$ *as the extended real-valued function $f^{**} : X^{**} \to \overline{\mathbb{R}}$ given by $f^{**} = (f^*)^*$.*

Note that, by construction, $f^* \in \Gamma(X^*, w^*)$ and $f^{**} \in \Gamma(X^{**}, w^{**})$.

**Proposition 1.2.10** *Let $(X, \theta)$ be a locally convex space and $f, g : X \to \overline{\mathbb{R}}$ be two extended real-valued functions. The following assertions hold:*

*(a)* $f^* \in \Gamma_0(X^*, w^*) \cup \{\underline{\omega}, \overline{\omega}\}$. *In particular, if $f \neq \overline{\omega}$ and if there exist $x^* \in X^*$ and $c \in \mathbb{R}$ such that $f \geq x^* + c$, then $f^*$ is proper.*

*(b)* $f(x) + f^*(x^*) \geq \langle x^*, x\rangle$, *for every $x \in X$ and every $x^* \in X^*$.*

*(c) If $f \leq g$, then $f^* \geq g^*$.*

*(d)* $f^{**}\big|_X \leq \overline{\operatorname{co}} f \leq \operatorname{cl}_\theta f \leq f$. *In particular,* $f^* = (\operatorname{cl}_\theta f)^* = (\overline{\operatorname{co}} f)^* = (f^{**}\big|_X)^*$.

*(e) If $\overline{\operatorname{co}} f$ is proper, then*

$$f^{**}\big|_X = \sup\{x^* + c \ : \ x^* \in X^*, \ c \in \mathbb{R} \textit{ such that } x^* + c \leq f\} = \overline{\operatorname{co}} f.$$

*Particularly, if $f \in \Gamma_0(X)$, then $f^{**}\big|_X = f$.*

*(f) For every $h \in \Gamma_0(X^*)$ we have that*

$$\left(h^*\big|_X\right)^* = h \iff h \textit{ is } w^*\textit{-lsc.}$$

**Proposition 1.2.11** *Let $(X, \theta)$ be a locally convex space. The following assertions hold:*

*(a) For a subset $S \subseteq X$ we have that*

$$(I_S)^* = \sigma_S \qquad \textit{and} \qquad (I_S)^{**} = (\sigma_S)^* = I_{\overline{\operatorname{co}}^{w^{**}}(S)}.$$

*(b) For two proper extended real-valued functions $f, g : X \to \overline{\mathbb{R}}$, we have that*

$$(f \Box g)^* = f^* + g^*.$$

*Moreover if $\overline{\operatorname{co}}(f \Box g)$ is proper, then*

$$(f^* + g^*)^*\big|_X = \overline{(\overline{\operatorname{co}} f) \Box (\overline{\operatorname{co}} g)}.$$

(c) *For two functions* $f, g \in \Gamma_0(X)$ *such that* $\mathrm{Cont}[f,\theta] \cap \mathrm{dom}\, g$ *is nonempty, we have that*

$$(f+g)^* = f^* \square g^*.$$

*Proof.* Statement $(a)$ is straight-forward following Definition 1.2.9. See [51, Corollary 2.3.5] for $(b)$, and [51, Theorem 2.8.3] for $(c)$. □

### 1.2.2 Moreau-Rockafellar Subdifferential

Before starting with the (Moreau-Rockafellar) subdifferential of an extended real-valued function, we need to recall the notion of Set-valued operators.

**Definition 1.2.12** *Let* $T$ *and* $S$ *two nonempty sets. We say* $M$ *is a* set-valued operator *from* $T$ *into* $S$*, which is denoted by* $M : T \rightrightarrows S$*, if for every* $t \in T$*,* $M(t)$ *is a (possible empty) subset of* $S$*.*

For a set-valued operator $M : T \rightrightarrows S$, we denote its *(effective) domain* and its *graph* as the sets

$$\mathrm{dom}\, M := \{t \in T \ : \ M(t) \neq \emptyset\} \quad \text{and} \quad \mathrm{gph}\, M := \{(t,s) \in T \times S \ : \ s \in M(t)\}.$$

We will usually identify the set-valued operator $M$ with its graph $\mathrm{gph}\, M$, that is, we will simply write $(t,s) \in M$ to denote the inclusion $s \in M(t)$. Also, for a second set-valued operator $R : T \rightrightarrows S$ we will write $M \subseteq R$ if $\mathrm{gph}\, M \subseteq R$ (or equivalently, if for every $t \in T$, $M(t) \subseteq R(t)$).

Finally, we denote by $M^{-1} : S \rightrightarrows T$ the inverse set-valued operator associated to $M$, that is, the set-valued operator given by $M^{-1}(s) := \{t \in T \ : \ s \in M(t)\}$.

When we work with subdifferentials (as we will see below during this section), we need some notion of continuity of set-valued operators. Here, we will present only two: *Outer semicontinuity* and *Upper semicontinuity.* For others notions of continuity, we refer the reader to [2].

**Definition 1.2.13** (Outer-semicontinuity) *Let* $(T,\tau), (S,\sigma)$ *be two Hausdorff topological spaces. A set-valued operator* $M : T \rightrightarrows S$ *is* $\tau$*-*$\sigma$*-outer semicontinuous (*$\tau$*-*$\sigma$*-osc, for short) at a point* $t_0 \in T$ *if for each net* $(t_i, s_i) \in M$ *which is* $\tau \times \sigma$*-convergent in* $T \times S$ *with* $t_i \to t_0$*, we have that*

$$\lim_i s_i \in M(t_0).$$

**Definition 1.2.14** (Upper-semicontinuity) *Let* $(T,\tau), (S,\sigma)$ *be two Hausdorff topological spaces. A set-valued operator* $M : T \rightrightarrows S$ *is* $\tau$*-*$\sigma$*-upper semicontinuous (*$\tau$*-*$\sigma$*-usc, for* *short) at a point* $t_0 \in T$ *if for every* $\sigma$*-open set* $U \subseteq S$ *containing* $M(t_0)$*, there exists a neighborhood* $V \in \mathcal{N}_T(t_0, \tau)$ *such that*

$$M(V) \subseteq U.$$

If there is no confusion, we will simply say that $M$ is osc (or usc) at some point, without writing the topologies involved. In general, outer semicontinuity and upper semicontinuity are not related, that is, no one implies the other. Nevertheless, under some qualification conditions, they are equivalent. The following propositions can be found in [2, Ch. 6 - §3].

Recall that a topological space $(S, \sigma)$ is said to be *regular* if for every $\sigma$-closed set $A \subseteq S$ and every point $s \in S \setminus A$, there exists two $\sigma$-open sets $U$ and $V$ such that $A \subseteq U$, $s \subseteq V$ and $U \cap V = \emptyset$.

**Proposition 1.2.15** *Let* $(T, \tau), (S, \sigma)$ *be two Hausdorff topological spaces,* $M : T \rightrightarrows S$ *be a set-valued operator and* $t_0 \in T$*. If* $(S, \sigma)$ *is regular and* $M(t_0)$ *is* $\sigma$*-closed, then*

$$M \text{ is usc at } t_0 \implies M \text{ is osc at } t_0.$$

**Proposition 1.2.16** *Let* $(T, \tau), (S, \sigma)$ *be two Hausdorff topological spaces,* $M : T \rightrightarrows S$ *be a set-valued operator and* $t_0 \in T$*. Assume that there exists a neighborhood* $W_0 \in \mathcal{N}_T(t_0, \tau)$ *such that* $\operatorname{cl}(M(W_0))$ *is* $\sigma$*-compact. Then,*

$$M \text{ is osc at } t_0 \implies M \text{ is usc at } t_0.$$

Now, let us get back to the locally convex space setting. We present the Moreau subdifferential of a function, which was introduced by Moreau and Rockafellar in the decade of 1960. The Moreau subdifferential is a set-valued operator that generalizes the notion of derivatives for convex functions. Even though it was first introduced for convex functions, we will defined it for general (not necessarily convex) ones, since we will use it in section 1.3 in the nonconvex case.

**Definition 1.2.17** (Moreau-Rockafellar Subdifferential) *Let* $(X, \theta)$ *be a locally convex space and* $f : X \to \mathbb{R}_\infty$ *be a proper extended real-valued function. For a point* $x_0 \in \operatorname{dom} f$ *we define* the (Moreau-Rockafellar) subdifferential of $f$ at $x_0$*, denoted by* $\partial f(x_0)$*, as the set of all functionals* $x^* \in X^*$ *satisfying*

$$\langle x^*, y - x_0 \rangle + f(x_0) \leq f(y), \quad \forall y \in X. \tag{1.8}$$

*Setting* $\partial f(x) := \emptyset$ *for all* $x \in X \setminus \operatorname{dom} f$*, the induced set-valued operator* $\partial f : X \rightrightarrows X^*$ *is called* the subdifferential of $f$.

Note that equation (1.8) holds for all $y \in X$ if and only if it holds for all $y \in \operatorname{dom} f$. The following propositions present the basic properties of the Moreau subdifferential. In what follows, we will always assume that $X$ is endowed with a locally convex topology $\theta$ and that $X^*$ denotes the dual space of $(X, \theta)$.

**Proposition 1.2.18** *Let $f : X \to \mathbb{R}_\infty$ be a proper function, $x \in \operatorname{dom} f$ and $x^* \in X^*$. Then,*

*(a)* $x^* \in \partial f(x) \Leftrightarrow \langle x^*, x\rangle \geq f(x) + f^*(x^*) \Leftrightarrow \langle x^*, x\rangle = f(x) + f^*(x^*)$.

*(b)* *$\partial f(x)$ is a (possible empty) convex $w^*$-closed subset of $X^*$.*

*(c)* *If $\partial f(x) \neq \emptyset$, then $\overline{\text{co}}\, f(x) = f(x)$ and $\partial(\overline{\text{co}}\, f)(x) = \partial f(x)$.*

*(d)* *If $f \in \Gamma_0(X)$, then $x^* \in \partial f(x) \Leftrightarrow x \in \partial f^*(x^*)$.*

*(e)* (Fermat rule) *A point $x \in X$ is a global minimum of $f$ (i.e., $x \in \operatorname{argmin} f$) if and only if $0 \in \partial f(x)$.*

**Example 1.2.19** Let $A$ be a convex closed set of $(X, \theta)$. The subdifferential of the indicator function $I_A$ is given by

$$\partial I_A(x) = \begin{cases} \{x^* \in X^* \ : \ \langle x^*, y - x\rangle \leq 0, \forall y \in A\} & \text{if } x \in A. \\ \emptyset & \text{otherwise.} \end{cases}$$

When $x \in A$, the set $\partial I_A(x)$ is clearly a convex $w^*$-closed cone of $X^*$ and it is commonly known as the *normal cone of $A$ at $x$*, also denoted as $N_A(x)$.

Now, let $K$ be a nomepty convex $w^*$-closed set of $X^*$. Noting that $(\sigma_K)^* = I_K$, Proposition 1.2.18(a) entails that the subdifferential of $\sigma_K$ is given by

$$\partial \sigma_K(x) = \{x^* \in K \ : \ \langle x^*, x\rangle = \sigma_K(x^*)\}. \tag{1.9}$$

This equation will play a fundamental role in Chapter 3.

**Proposition 1.2.20** *Let $f : X \to \mathbb{R}_\infty$ be a proper convex function and $x \in \operatorname{dom} f$. If $x \in \operatorname{Cont}[f, \theta]$, then $\partial f(x)$ is nonempty.*

**Proposition 1.2.21** *Let $f : X \to \mathbb{R}_\infty$ be a proper convex function and let $x \in \operatorname{Cont}[f, \theta]$. Then, $\partial f$ is locally $w^*$-bounded near $x$, which means that there exists a $\theta$-neighborhood $U$ of zero such that $\partial f(x + U)$ is $w^*$-bounded. Even more, $U$ can be chosen such that $\partial f(x + U)$ is contained in an absolutely convex (convex and balanced) $w^*$-compact set of $X^*$.*

*Proof.* In this proof, the topological notation will refer to the $\theta$ topology for $X$. Since $f$ is continuous at $x$, there exists a convex symmetric neighborhood $U$ of zero such that, $f$ is Lipschitz continuous on $x + 2U$ with constant $\gamma > 0$. Then, for $z \in x + U$ and $z^* \in \partial f(z)$

we have that for all $h \in U$,

$$|\langle z^*, h\rangle| = \max(\langle z^*, h\rangle, \langle z^*, -h\rangle) \leq \max(|f(z+h) - f(z)|, |f(z-h) - f(z)|) \leq \gamma\rho_{2U}(h).$$

Then, because $\rho_{2U}$ and $|\langle z^*, \cdot\rangle|$ are positively homogeneous,

$$z^* \in B = \{x^* \in X^* \mid |\langle x^*, h\rangle| \leq \gamma\rho_{2U}(h), \ \forall h \in X\},$$

and so $\partial f(x+U) \subseteq B$. It is easy to verify that $B = (2\gamma^{-1}U)^o$ and so, by the Alaoglu-Bourbaki theorem, $B$ is $w^*$-compact. From the construction, $B$ is also absolutely convex, and so the proof is finished. □

**Proposition 1.2.22** *Let $f, g : X \to \mathbb{R}_\infty$ be two proper functions. Then,*

*(a) For every $\lambda > 0$ and every $x \in X$, $\partial(\lambda f)(x) = \lambda\partial f(x)$.*

*(b) For all $x \in X$, $\partial f(x) + \partial g(x) \subseteq \partial(f+g)(x)$.*

*(c) If $f, g \in \Gamma_0(X)$ and $\operatorname{Cont}[f, \theta] \cap \operatorname{dom} g \neq \emptyset$, then $\partial f(x) + \partial g(x) = \partial(f+g)(x)$.*

*(d) For $x \in X$ and $x_1, x_2 \in X$ satisfying $x = x_1 + x_2$, then*

$$(f\Box g)(x) = f(x_1) + g(x_2) \implies \partial(f\Box g)(x) = \partial f(x_1) \cap \partial g(x_2).$$

*Conversely, if $\partial f(x_1) \cap \partial g(x_2) \neq \emptyset$, then the inf-convolution at $x$ verifies $(f\Box g)(x) = f(x_1) + g(x_2)$.*

*Proof.* Properties $(a)$ and $(b)$ are straight-forward from Definition 1.2.17. See [51, Theorem 2.8.3] for $(c)$, and [51, Corollary 2.4.7] for $(d)$.

□

**Definition 1.2.23** *Let $(X, \theta)$ be a locally convex space, $f : X \to \mathbb{R}_\infty$ be a proper extended real-valued function and $\varepsilon \geq 0$. For a point $x_0 \in \operatorname{dom} f$ we define the* $\varepsilon$-subdifferential of $f$ at $x_0$, *denoted by $\partial_\varepsilon f(x_0)$, as the set of all functionals $x^* \in X^*$ satisfying*

$$\langle x^*, y - x_0\rangle + f(x_0) \leq f(y) + \varepsilon, \quad \forall y \in X. \tag{1.10}$$

*Setting $\partial_\varepsilon f(x) := \emptyset$ for all $x \in X \setminus \operatorname{dom} f$, the induced set-valued operator $\partial_\varepsilon f : X \rightrightarrows X^*$ is called* the $\varepsilon$-subdifferential of $f$.

From the latter definition, it is clear that of $\varepsilon = 0$, $\partial_0 f = \partial f$. Also, the following proposition provides some basic properties of the $\varepsilon$-subdifferential.

**Proposition 1.2.24** *Let $f : X \to \mathbb{R}_\infty$ be a proper extended real-valued function, $x \in \operatorname{dom} f$, $x^* \in X^*$ and $\varepsilon > 0$. We have that* *(a)* $\partial f(x) = \bigcap_{\delta>0} \partial_\delta f(x)$ *and* $\partial_\varepsilon f(x) = \bigcap_{\delta>\varepsilon} f(x)$.

*(b)* $x^* \in \partial_\varepsilon f(x) \Leftrightarrow f(x) + f^*(x^*) \leq \varepsilon + \langle x^*, x\rangle \Rightarrow x^* \in \partial_\varepsilon f^*(x^*)$.

*(c)* $\partial_\varepsilon f(x)$ *is convex* $w^*$*-closed set of* $X^*$.

*(d) If* $f \in \Gamma_0(X)$, *then* $\partial_\varepsilon f(x)$ *is nonempty and*

$$x^* \in \partial_\varepsilon f(x) \iff x \in \partial_\varepsilon f^*(x^*).$$

*(e) If* $f \in \Gamma_0(X)$ *and* $x \in \mathrm{Cont}[f, \theta]$, *then* $\partial_\varepsilon f(x)$ *is* $w^*$*-compact.*

*(f)* (Fermat Rule) $x$ *is an* $\varepsilon$*-minimum of* $f$ *(i.e.,* $f(x) \leq \inf_X f + \varepsilon$*) if and only if* $0 \in \partial_\varepsilon f(x)$.

The relationship between the differentiability of a convex function and the subdifferential can be formulated in locally convex spaces in terms of the *directional derivative.* We will return on this subject in Chapter 3.

**Definition 1.2.25** (Directional derivative) *Let* $f : X \to \mathbb{R}_\infty$ *a proper extended real-valued function and let* $x \in \operatorname{dom} f$. *We say that* $f$ is directionally differentiable at $x$ *if for every* $h \in X$ *the limit*

$$f'(x;h) := \lim_{t \searrow 0} \frac{f(x+th) - f(x)}{t}$$

*exists in* $\overline{\mathbb{R}}$. *The extended real-valued function* $f'(x;\cdot) : X \to \overline{\mathbb{R}}$ *is called* the directional derivative of $f$ at $x$.

**Proposition 1.2.26** *Let* $s : X \to \mathbb{R}_\infty$ *be a proper sublinear function. Then, for each* $x \in X$ *we have that*

$$\partial s(x) = \{x^* \in X^* \;:\; x^* \in \partial s(0) \text{ and } \langle x^*, x\rangle = s(x)\}.$$

*Moreover, if* $s \in \Gamma_0(X)$ *then for each* $x \in X$ *we have that*

$$s(x) = \sup\{\langle x^*, x\rangle \;:\; x^* \in \partial s(0)\} = \sigma_{\partial s(0)}(x).$$

**Proposition 1.2.27** *Let* $f : X \to \mathbb{R}_\infty$ *be a proper convex function and let* $x \in \operatorname{dom} f$. *Then,*

*(a)* $f$ *is directionally differentiable at* $x$ *and* $f'(x;\cdot)$ *is sublinear. Moreover,* $f'(x;\cdot)$ *verifies*

$$f'(x;h) = \inf_{t>0} \frac{f(x+th) - f(x)}{t}, \qquad \forall h \in X.$$

*(b) We always have* $\partial f(x) = \partial(f'(x;\cdot))(0)$. *Also,* $\partial f(x) \neq \emptyset$ *if and only if* $f'(x;\cdot)$ *is lsc at* $0$. *In such a case,*

$$\overline{f'(x;\cdot)}(h) = \sup\{\langle x^*, h\rangle \ : \ x^* \in \partial f(x)\} = \sigma_{\partial f(x)}(h), \quad \forall h \in X.$$

*Moreover, if* $x \in \operatorname{Cont}[f,\theta]$, *then* $f'(x;\cdot)$ *is finite,* $\theta$-$\tau_0$*-continuous and it coincides with* $\sigma_{\partial f(x)}$.

# 1.3 Integration Formulas and SDPD spaces

We are now able to present the principal results of this chapter (presented in the Introduction of Part I), namely, the integration formula for nonconvex functions using the Moreau-Rockafellar subdifferential in the locally convex space setting. To do so, we will introduce a suitable notion of epi-pointedness and we will show some results concerning the subdifferential of continuous functions, which are known to be true in the Banach space setting but they seem to be new in our context.

## 1.3.1 Preliminary results on continuous functions

For any function $f \in \Gamma_0(X,\theta)$, the $\theta$-continuity points play a fundamental role. We already saw in Propositions 1.2.20, 1.2.21 and 1.2.27, that if $x \in \operatorname{Cont}[f,\theta]$, then $\partial f(x)$ is nonempty and $w^*$-compact and

$$f'(x;u) = \max\{\langle x^*, u\rangle \ : \ x^* \in \partial f(x)\}, \ \forall u \in X. \tag{1.11}$$

Also, by Proposition 1.2.7, if $\operatorname{Cont}[f,\theta]$ is nonempty, then $\operatorname{Cont}[f,\theta] = \operatorname{int}_\theta(\operatorname{dom} f)$. In view of sections 1.3.2 and 1.3.3, we will study here some results on $\theta$-continuity and subdifferentiability that are well-known in the Banach space setting.

Recall that a set-valued operator $M : X \rightrightarrows X^*$ is *monotone* provided that

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq 0, \quad \forall (x_1, x_1^*), (x_2, x_2^*) \in M. \tag{1.12}$$

When in addition there is no different set-valued monotone operator whose graph contains $\operatorname{gph} M$, one says that $M$ is *maximal monotone*.

**Lemma 1.3.1** *Let* $f \in \Gamma_0(X,\theta)$ *with* $D = \operatorname{Cont}[f,\theta] \neq \emptyset$. *Then, for each* $x \in \overline{\operatorname{dom} f}^{\theta}$ *we have that*

$$f(x) = \liminf_{D \ni y \to x} f(y).$$

*Proof.* In this proof, the topological notation will refer to the $\theta$ topology for $X$. We know that for $x \in \overline{\operatorname{dom} f}$,

$$f(x) = \liminf_{y \to x} f(y) = \liminf_{\operatorname{dom} f \ni y \to x} f(y).$$

On the other hand, since $\operatorname{Cont}[f]$ is nonempty, the epigraph of $f$ has nonempty interior in the $\theta \times \tau_0$ topology, and so

$$\operatorname{epi} f = \overline{\operatorname{int}(\operatorname{epi} f)}.$$

Let us consider $f\big|_D$. It is direct that $\operatorname{int}(\operatorname{epi} f) \subseteq \operatorname{epi} f\big|_D \subseteq \operatorname{epi} f$, thus $\overline{\operatorname{epi} f\big|_D} = \operatorname{epi} f$. So, there exists a net $(x_i, r_i)_{i \in I} \subseteq \operatorname{epi} f\big|_D$ such that $(x_i, r_i) \to (x, f(x))$. Thus,

$$f(x) = \liminf_{\operatorname{dom} f \ni y \to x} f(y) \leq \liminf_{D \ni y \to x} f(y) \leq \liminf_{i \in I} f(x_i) \leq \liminf_{i \in I} r_i = f(x),$$

and so $f(x) = \liminf_{D \ni y \to x} f(y)$. □

The next lemma is related to maximal monotone operators. The result is well-known when $(X, \theta)$ is a Banach space, and it was proved in general locally convex spaces by Moreau [34]. We rediscovered it in [18].

**Lemma 1.3.2** *Let $f \in \Gamma_0(X, \theta)$ with $\operatorname{Cont}[f, \theta] \neq \emptyset$. Then $\partial f$ is a maximal monotone operator.*

*Proof.* We follow the proof in [36, Theorem 2.25], where $f$ is assumed to be continuous in all of $X$ and $X$ is a Banach space. Let $(y, y^*) \in X \times X^*$ such that $y^* \notin \partial f(y)$. In order to prove that $\partial f$ is maximal monotone, it is sufficient to show that there exists some $(x, x^*) \in \partial f$ such that

$$\langle x^* - y^*, x - y \rangle < 0.$$

Without loss of generality, we can suppose that $(y, y^*) = (0, 0)$. If not, we can replace $f$ with the function $g \in \Gamma_0(X, \theta)$ given by $g(x) = f(x + y) - \langle y^*, y \rangle$ and it is easily verified that

$$\partial g(x) = \partial f(x + y) - y^*, \ \forall x \in X.$$

Since $0 \notin \partial f(0)$, we know that $0$ is not a global minimum of $f$. Then, there exists $x \in \operatorname{dom} f$ such that $f(x) < f(0)$. This and Lemma 1.3.1 guarantee that there exists $x_1 \in \operatorname{Cont}[f]$ such that $f(x_1) < f(0)$. Consider now the function $h : [0, 1] \subset \mathbb{R} \to \mathbb{R}$ given by $h(t) = f(tx_1)$. For $\lambda \in (0, 1)$ we have that

$$\begin{aligned} h'(\lambda; 1) &= \lim_{t \searrow 0} \frac{h(\lambda + t) - h(\lambda)}{t} \\ &= \lim_{t \searrow 0} \frac{f(\lambda x_1 + t x_1) - f(\lambda x_1)}{t} = f'(\lambda x_1; x_1). \end{aligned}$$

Then, because $h(1) = f(x_1) < f(0) = h(0)$, there exists some $t_1 \in (0,1)$ such that $h'(t_1; 1) < 0$. Fixing $x = t_1 x_1$ and noting that $x \in \operatorname{int}(\operatorname{dom} f) = \operatorname{Cont}[f]$, we have that

$$\max\{\langle x^*, x\rangle \mid x^* \in \partial f(x)\} = t_1 f'(x; x_1) < 0,$$

and hence there is an $x^* \in \partial f(x)$ such that $\langle x^*, x\rangle \leq t_1 f'(x; x_1) < 0$, which completes the proof. □

The last interesting property of the subdifferential of a proper convex function $f$ is its outer-semicontinuity at continuity points of $f$.

**Proposition 1.3.3** *Let $f : X \to \mathbb{R}_\infty$ be a proper convex function with* $\operatorname{Cont}[f, \tau(X, X^*)] \neq \emptyset$*. Then $\partial f$ is $\tau(X, X^*)$-$w^*$-outer semicontinuous at each point of* $\operatorname{Cont}[f, \tau(X, X^*)]$.

*Proof.* In this proof, the topological notation will refer to the Mackey topology $\tau(X, X^*)$ for $X$ and the $w^*$-topology for $X^*$. Fix $x \in \operatorname{Cont}[f]$ and let $(x_i, x_i^*)_{i\in I} \subseteq \partial f$ be a convergent net with limit $(x, x^*) \in X \times X^*$. Applying Proposition 1.2.21, there exists an open neighborhood $V$ of $x$ such that $\partial f(V)$ is contained in an absolutely convex $w^*$-compact set $K$ of $X^*$. Thus, there exists an $i_0 \in I$ such that for all $i \geq i_0$, $x_i \in V$, and so $(x_i^*)_{i\geq i_0} \subseteq K$. Now, fix $y \in X$. We have that

$$\langle x_i^*, y - x_i\rangle = \langle x_i^*, y - x\rangle + \langle x_i^*, x - x_i\rangle \leq \langle x_i^*, y - x\rangle + \sup_{z^*\in K}\langle z^*, x - x_i\rangle.$$

It is clear that $\langle x_i^*, y - x\rangle \to \langle x^*, y - x\rangle$. Also, since $(x_i - x)_{i\geq i_0}$ is convergent to zero, by definition of the Mackey topology (see Remark 1.1.10), it converges to zero uniformly over absolutely convex $w^*$-compact sets of $X^*$. In particular, we have that

$$\lim_{i\geq i_0} \sup_{z^*\in K}\langle z^*, x_i - x\rangle = 0.$$

Then, provided $f$ is continuous at $x$, we have that

$$0 \leq f(y) - f(x_i) - \langle x_i^*, y - x_i\rangle \to f(y) - f(x) - \langle x^*, y - x\rangle,$$

and so $f(x) + \langle x^*, y - x\rangle \leq f(y)$. Since $y$ is arbitrary, $x^* \in \partial f(x)$. This finishes the proof. □

**Remark 1.3.4** The above Proposition replaces the well-known norm-$w^*$-upper semicontinuity property of $\partial f$, when $X$ is a Banach space (see, e.g., [36, Proposition 2.5]). In fact, combining Propositions 1.2.21 and 1.3.3 we can derive that $\partial f$ is also $\tau(X, X^*)$-$w^*$-upper semicontinuous at each point of $\operatorname{Cont}[f, \tau(X, X^*)]$, since $\partial f$ meets the hypothesis of Proposition 1.2.16. Even though we won't use it directly, this last result is worth mentioning, since upper-semicontinuity has been an extremely useful property of the subdifferential in the development of convex analysis. Here, the use of Mackey topology $\tau(X, X^*)$ is fundamental.

This property can also be found in [34], and we also rediscovered it in [18]. Moreau proved directly the upper-semicontinuity, while we proved first the outer-semicontinuity and derived the upper-semicontinuity by the local $w^*$-compactness of the subdifferential.

### 1.3.2 Main Results: SDPD functions and Integration Theorem

The first concept we need is an appropriate extension of the notion of epi-pointedness:

**Definition 1.3.5** ($\tau$-epi-pointedness) *Let $\tau$ be a locally convex topology on $X^*$ finer than the $w^*$-topology. A function over $X$ with values in $\mathbb{R}_\infty$ is said to be $\tau$-epi-pointed if the set of $\tau$-continuity points of its conjugate is nonempty.*

Outside the RNP setting, epi-pointedness and lower semicontinuity are not enough to ensure the nonemptyness of the subdifferential, and therefore it is not possible to perform any type of integration as we want to, having only these hypothesis.

**Proposition 1.3.6** *Let $X$ be a Banach space which lacks the RNP. Then there exists an epi-pointed lower semicontinuous function $f : X \to \mathbb{R}_\infty$ such that $\partial f(x) = \emptyset$, for all $x \in X$.*

*Proof.* Since $X$ is a space lacking the RNP, then, $X \times \mathbb{R}$ also lacks the RNP. Applying [9, Theorem 3.7.8], there exists an equivalent norm $p$ over $X \times \mathbb{R}$ and a closed set $D \subseteq \operatorname{int}\left(\mathbb{B}_{(X\times\mathbb{R},p)}\right)$ (where $\mathbb{B}_{(X\times\mathbb{R},p)}$ denotes the unit ball in $X \times \mathbb{R}$ given by $p$), such that

$$\mathbb{B}_{(X\times\mathbb{R},p)} = \overline{\operatorname{co}}(D).$$

Let us consider the set $E = \{(x,\lambda) \;:\; \exists \alpha \leq \lambda,\ (x,\alpha) \in D\}$. Evidently, $E$ is the epigraph of the function

$$\begin{aligned} f : X &\to \mathbb{R}_\infty \\ x &\mapsto \inf\{t \in \mathbb{R} \;:\; (x,t) \in E\}. \end{aligned}$$

We claim that $f$ is the function that we are looking for. First, we show the epi-pointedness. Note that, since $\overline{\operatorname{co}}(D) = \mathbb{B}_{(X\times\mathbb{R},p)}$, we have

$$\overline{\operatorname{co}}(E) = \{(x,\lambda) \;:\; \exists \alpha \leq \lambda,\ p(x,\alpha) \leq 1\},$$

and so $\overline{\operatorname{co}}\, f$ is given by

$$\overline{\operatorname{co}}\, f(x) = \inf\{t \in \mathbb{R} \;:\; p(x,t) \leq 1\}.$$

Then we have that

$$\begin{aligned} f^*(x^*) &= \sup_{x \in X} \left\{\langle x^*, x\rangle - \inf\{t \in \mathbb{R} \;:\; p(x,t) \leq 1\}\right\} \\ &= \sup_{(x,t)\in X\times\mathbb{R}} \left\{\langle x^*, x\rangle - \Big(t + I_{\mathbb{B}_{(X\times\mathbb{R},p)}}(x,t)\Big)\right\} \\ &= p_*(x^*, -1), \end{aligned}$$

where $p_*$ denotes the dual norm on $X^* \times \mathbb{R}$ induced by $p$. We see that $f^*$ is continuous and real-valued, which proves that $f$ is epi-pointed. Let us suppose now, by contradiction, that $f$ is not lower semicontinuous. Then, there exists a sequence $(x_n) \in \operatorname{dom} f$ converging to some point $x \in X$ such that

$$\bar{t} = \liminf f(x_n) < f(x).$$

We may assume that $(f(x_n))$ is converging to $\bar{t}$. Since $(x_n) \subset \operatorname{dom} f$, we have that $(x_n, f(x_n)) \subset D$ and $(x, \bar{t}) = \lim(x_n, f(x_n)) \in D$. Then we have that

$$f(x) = \inf\{t \in \mathbb{R} \ : \ \exists \alpha \leq t,\ (x, \alpha) \in D\} \leq \bar{t} < f(x),$$

which is clearly a contradiction. It only remains to verify the emptyness of $\partial f(x)$, for $x \in \operatorname{dom} f$. Let $x \in \operatorname{dom} f$. Since $(x, f(x)) \in D$, $(x, \overline{\operatorname{co}} f(x)) \in \mathbb{S}_{(X\times\mathbb{R},p)}$ (where $\mathbb{S}_{(X\times\mathbb{R},p)}$ denotes the unit sphere in $X \times \mathbb{R}$ given by $p$) and $D \cap \mathbb{S}_{(X\times\mathbb{R},p)} = \emptyset$, we have that $\overline{\operatorname{co}} f(x) < f(x)$, and so $\partial f(x) = \emptyset$. □

The next definition provides the sufficient conditions needed to perform integration of a function using the same techniques applied in [17], as it will be shown in Theorem 1.3.12.

**Definition 1.3.7** (SDPD function) *We say that a function $f : X \to \mathbb{R}_\infty$ is Subdifferential Dense Primal Determined (SDPD) if it is $\tau(X^*, X^{**})$-epi-pointed and the set of functionals*
*$x^* \in \operatorname{Cont}[f^*, \tau(X^*, X^{**})]$ which satisfy the equality*

$$\partial f^*(x^*) = \overline{\operatorname{co}}^{w^{**}} \left[(\partial f)^{-1}(x^*)\right] \tag{1.13}$$

*is $\tau(X^*, X^{**})$-dense in $\operatorname{Cont}[f^*, \tau(X^*, X^{**})]$.*

The choice of the Mackey topology is crucial for three reasons: First, we have more epi-pointed functions; Second, in the Banach spaces setting, the Mackey topology $\tau(X^*, X^{**})$ coincides with the norm topology in $X^*$; And finally, the subdifferential of any conjugate function is $\tau(X^*, X^{**})$-$w^{**}$-outer semicontinuous at each point of $\tau(X^*, X^{**})$-continuity, according to Proposition 1.3.3. This final property will be a key to prove the integration theorem. The problem with this choice is that the density of the functionals which satisfy equality (1.13) is harder: We will need the equation to hold at more points.

**Lemma 1.3.8** *Let $f, h \in \Gamma_0(X, \theta)$, both having at least one point of $\tau(X, X^*)$-continuity and satisfying the following two conditions:*

1. *$D := \operatorname{Cont}[f, \tau(X, X^*)] = \operatorname{Cont}[h, \tau(X, X^*)]$.*

2. *$\partial f(x) \subseteq \partial h(x)$, for all $x \in D$.*

*Then $f$ and $h$ are equal up to an additive constant.*

*Proof.* In this proof, the topological notation will refer to the Mackey topology $\tau(X, X^*)$ for $X$. Without loss of generality, we may suppose that $0 \in D$. Fix then $x \in D$ and define the functions $\varphi_f, \varphi_h : \mathbb{R} \to \mathbb{R}_\infty$ given by

$$\varphi_f(t) = f(tx) \quad \text{and} \quad \varphi_h(t) = h(tx).$$

It is clear that $\varphi_f, \varphi_h \in \Gamma_0(\mathbb{R})$ and that both are continuous in $[0, 1]$. Defining the linear continuous operator $A : \mathbb{R} \to X$ given by $A(t) = tx$, we have that $\varphi_f = f \circ A$ and $\varphi_h = h \circ A$. Since $0 \in D$, the subdifferential chain rule holds and then for all $t \in [0, 1]$,

$$\partial\varphi_f(t) = \langle \partial f(tx), x\rangle \subseteq \langle \partial h(tx), x\rangle = \partial\varphi_h(t).$$

Then, again because of the continuity of $\varphi_f$ and $\varphi_h$, the subdifferential sum rule holds and then, for all $t \in \mathbb{R}$,

$$\begin{aligned}\partial\left(\varphi_f + I_{[0,1]}\right)(t) &= \partial\varphi_f(t) + \partial I_{[0,1]}(t)\\ &\subseteq \partial\varphi_h(t) + \partial I_{[0,1]}(t)\\ &= \partial\left(\varphi_h + I_{[0,1]}\right)(t).\end{aligned}$$

Thus, by the classical results of integration in real analysis, we conclude that $\varphi_f$ and $\varphi_h$ are equal in the interval $[0, 1]$ up to an additive constant. In particular,

$$f(x) - f(0) = \varphi_f(1) - \varphi_f(0) = \varphi_h(1) - \varphi_h(0) = h(x) - h(0).$$

Fixing $c = f(0) - h(0)$, we have $f(x) = h(x) + c$ for all $x \in D$. To finish the proof, consider $x \in \overline{D}$ $\left(= \overline{\operatorname{dom} f} = \overline{\operatorname{dom} h}\right)$. Applying Lemma 1.3.2 we obtain

$$\begin{aligned}f(x) &= \liminf_{y\to x} f(y)\\ &= \liminf_{D\ni y\to x} f(y) = \liminf_{D\ni y\to x} h(x) + c\\ &= \liminf_{y\to x} h(y) + c = h(x) + c.\end{aligned}$$

Noting that the equality is trivial at points $x \in X \setminus \overline{D}$, the proof is concluded. □

**Lemma 1.3.9** *Let $f, g \in \Gamma_0(X, \theta)$ and $V \subseteq \operatorname{Cont}[f, \tau(X, X^*)] \cap \operatorname{Cont}[g, \tau(X, X^*)]$ be a nonempty $\tau(X, X^*)$-open set. If there exists a $\tau(X, X^*)$-dense subset $D \subseteq V$ for which*

$$\partial f(x) \subseteq \partial g(x), \quad \text{for all } x \in D,$$

*then $\partial f(x) \subseteq \partial g(x)$, for all $x \in V$.*

*Proof.* In this proof, the topological notation will refer to the Mackey topology $\tau(X, X^*)$ for $X$. Let us suppose that there exist $x \in V$ and $x^* \in X^*$ such that $x^* \in \partial f(x) \setminus \partial g(x)$. Since $\partial g(x)$ is $w^*$-closed and nonempty, we can apply the separation theorem with the $w^*$-topology on $X^*$ to obtain $z \in X \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle x^*, z\rangle < \alpha < \langle u^*, z\rangle, \text{ for all } u^* \in \partial g(x).$$

Consider then the $w^*$-open set $W = \{u^* \in X^* : \langle u^*, z\rangle > \alpha\}$, which contains $\partial g(x)$, and the sequence $(z_n) \subseteq X$ given by

$$z_n = x - \frac{1}{n}z.$$

Without loss of generality, we may assume that $(z_n) \subseteq V$. We will show first that there exists an $n_0 \in \mathbb{N}$ such that $\partial g(z_{n_0}) \subseteq W$. If not, we choose $z_n^* \in \partial g(z_n)\backslash W$ for each $n \in \mathbb{N}$. Since $\partial g$ is locally $w^*$-bounded in $V$ (see Proposition 1.2.21), it is direct that $\{z_n^* : n \in \mathbb{N}\}$ is a $w^*$-bounded set, and so $(z_n^*)$ has a $w^*$-convergent subnet $(z_{\varphi(i)}^*)_{i\in I}$. Since $(z_n)$ converges to $x$, so does $(z_{\varphi(i)})_{i\in I}$. Thus, from the $\tau(X,X^*)$-$w^*$-outer semicontinuity of $\partial g$ at $x$ (see Proposition 1.3.3), we have that

$$w^*\text{-}\lim z_{\varphi(i)}^* \in \partial g(x) \subseteq W,$$

which is clearly a contradiction. Fix then $y = z_{n_0}$ such that $\partial g(y) \subseteq W$. Because of the density of $D$, there exists a net $(y_i) \subseteq D$ converging to $y$ and a net $(y_i^*) \subseteq X^*$ with $y_i^* \in \partial f(y_i)$. By Proposition 1.2.21 again, we can assume that $(y_i^*)_{i\in I}$ is included in a $w^*$-compact set of $X^*$. In particular, $(y_i^*)_{i\in I}$ has a $w^*$-convergent subnet, that we will continue denoting by $(y_i^*)_{i\in I}$. We have then $(y_i, y_i^*) \in \partial f \cap \partial g$ and since both subdifferentials are $\tau(X,X^*)$-$w^*$-outer semicontinuous at $y$, we deduce

$$y^* = w^*\text{-}\lim y_i^* \in \partial f(y) \cap \partial g(y).$$

Then, $y^* \in W$ and

$$\langle x^* - y^*, x - y\rangle = \frac{1}{n_0}\langle x^* - y^*, z\rangle < 0,$$

which is a contradiction with the monotonicity of $\partial f$. □

**Lemma 1.3.10** *Let $(T,\tau)$ be a Hausdorff topological space, $f : T \to \mathbb{R}_\infty$ be a proper function and $V \subseteq \operatorname{dom} f$ be a $\tau$-open set. If there exists a $\tau$-dense subset $D \subseteq V$ for which*

$$\limsup_{D\ni s\to t} f(s) = f(t),\ \forall t \in V,$$

*then $f$ is $\tau$-upper semicontinuous in $V$.*

*Proof.* Let us fix $\bar{t} \in V$ and a net $(t_i)_{i\in I} \subseteq T$ converging to $\bar{t}$. Without loss of generality, $(t_i)_{i\in I} \subseteq V$. Then, for each $i \in I$, there exists a net $(t_{i,j})_{j\in J(i)} \subseteq D$ converging to $t_i$ with

$$\limsup_j f(t_{i,j}) = f(t_i).$$

Fix $\varepsilon > 0$ and consider the set of indexes $\Lambda$ given by the tuples $(i,j,W)$ such that

1. $j \in J(i)$ and $W$ is an open set with $\bar{t} \in W \subseteq V$.

2. $t_{i,j} \in W$.

3. $f(t_{i,j}) \geq f(t_i) - \varepsilon$.

Consider the following preorder in $\Lambda$: $(i_1, j_1, W_1) \prec (i_2, j_2, W_2)$ if $i_1 \leq i_2$ and $W_1 \supseteq W_2$, in the case that $i_1 \neq i_2$, or if $j_1 \leq j_2$ and $W_1 \supseteq W_2$, in the case that $i_1 = i_2$.

We claim that the net $(t_\alpha)_{\alpha\in\Lambda}$, where $t_{(i,j,W)} := t_{i,j}$ for each $(i, j, W) \in \Lambda$, converges to $\bar{t}$. Let $W$ be an open set with $\bar{t} \in W \subseteq V$. Since $t_i \to \bar{t}$, there exists $i_0 \in I$ such that, for each $i \geq i_0$, $t_i \in W$. In particular, $t_{i_0} \in W$ and, since $W$ is an open set, we have that $W \in \mathcal{N}(t_{i_0})$. Then, provided

$$\limsup_{j\in J(i_0)} f(t_{i_0,j}) = f(t_{i_0}),$$

there exists $j_0 \in J(i_0)$ such that $t_{i_0,j_0} \in W$ and $f(t_{i_0,j_0}) \geq f(t_{i_0}) - \varepsilon$. Let $\alpha_0 = (i_0, j_0, W)$ which, by construction, belongs to $\Lambda$. For each $\alpha = (i', j', W') \in \Lambda$ such that $\alpha_0 \prec \alpha$, we get that $t_\alpha \in W' \subseteq W$, and so, the net $(t_\alpha)_{\alpha\in\Lambda}$ is eventually in $W$. Since $W$ is an arbitrary open neighborhood of $\bar{t}$, we conclude that $(t_\alpha)_{\alpha\in\Lambda}$ converges to $\bar{t}$, as we claimed. In particular, we have that

$$\limsup_{\alpha} f(t_\alpha) \leq \limsup_{D\ni t\to\bar{t}} f(t) = f(\bar{t}).$$

Also, noting that $\limsup_{\alpha} f(t_\alpha) \geq \limsup_{i} f(t_i) - \varepsilon$ we deduce

$$\limsup_{i} f(t_i) - \varepsilon \leq f(\bar{t}).$$

Thus, because $\varepsilon$ is arbitrary, we conclude that $\limsup_{i} f(t_i) \leq f(\bar{t})$, which finishes the proof. $\square$

Note that Lemma 1.3.10 remains true if we replace upper semicontinuity by lower semicontinuity (and lim sup by lim inf): Just apply the same proof to $-f$ instead of $f$. Thus, it also remains true if we replace upper semicontinuity by continuity (and lim sup by lim).

**Proposition 1.3.11** *Let $f : X \to \mathbb{R}_\infty$ be an SDPD function and $g : X \to \mathbb{R}_\infty$ be any function satisfying*

$$\partial f(x) \subseteq \partial g(x), \ \forall x \in X.$$

*Then,* $\mathrm{Cont}[f^*, \tau(X^*, X^{**})] \subseteq \mathrm{Cont}[g^*, \tau(X^*, X^{**})]$.

*Proof.* Endow $X^*$ with the Mackey topology $\tau(X^*, X^{**})$. Since $f$ is SDPD, there exists a dense subset $D \subseteq \mathrm{Cont}[f^*]$ such that the equality (1.13) holds for each $x^* \in D$. In particular, we have that for each $x^* \in D$

$$\partial f^*(x^*) \subseteq \overline{\mathrm{co}}^{w^{**}} \left[(\partial f)^{-1}(x^*)\right] \subseteq \overline{\mathrm{co}}^{w^{**}} \left[(\partial g)^{-1}(x^*)\right] \subseteq \partial g^*(x^*).$$

Fix $z^* \in \mathrm{Cont}[f^*]$ and let $(z^*_\alpha) \subseteq D$ be a net converging to $z^*$. Consider $(z_\alpha) \subseteq X$ such that $z_\alpha \in X \cap \partial f^*(z^*_\alpha)$. Since $f^*$ is continuous at $z^*$, we can assume that $(z_\alpha)$ is bounded (see Proposition 1.2.21) and so there exist $z^{**} \in X^{**}$ and a subnet $(z_{\varphi(i)})_{i\in I}$ such that $z_{\varphi(i)} \rightharpoonup^{w^{**}} z^{**}$. Since $z_{\varphi(i)} \in X \cap \partial g^*(z^*_{\varphi(i)})$ we have that for all $y^* \in X^*$

$$\langle z_{\varphi(i)}, y^* - z^*_{\varphi(i)}\rangle + g^*(z^*_{\varphi(i)}) \leq g^*(y^*).$$

Define $K = \{-z_{\varphi(i)}, z_{\varphi(i)} \ : \ i \in I\}$, which is a bounded set. Since $(z_{\varphi(i)})$ $w^{**}$-converges to $z^{**}$ and $(z^*_{\varphi(i)})$ converges to $z^*$, we have that

$$\langle z_{\varphi(i)}, y^* - z^*\rangle \to \langle z^{**}, y^* - z^*\rangle$$

$$|\langle z_{\varphi(i)}, z^* - z^*_{\varphi(i)}\rangle| \leq \sup_{z\in K}\langle z, z^* - z^*_\alpha\rangle \to 0.$$

Thus, we deduce

$$\langle z_{\varphi(i)}, y^* - z^*_{\varphi(i)}\rangle = \langle z_{\varphi(i)}, y^* - z^*\rangle + \langle z_{\varphi(i)}, z^* - z_{\varphi(i)}\rangle \to \langle z^{**}, y^* - z^*\rangle.$$

Finally, we have that

$$\begin{aligned}\langle z^{**}, y^* - z^*\rangle + g^*(z^*) &\leq \langle z^{**}, y^* - z^*\rangle + \liminf_i g^*(z^*_{\varphi(i)})\\ &= \liminf_i \langle z_{\varphi(i)}, y^* - z^*_{\varphi(i)}\rangle + \liminf_i g^*(z^*_{\varphi(i)})\\ &\leq \liminf_i [\langle z_{\varphi(i)}, y^* - z^*_{\varphi(i)}\rangle + g^*(z^*_{\varphi(i)})]\\ &\leq g^*(y^*).\end{aligned}$$

Then $z^{**} \in \partial g^*(z^*)$ and so, $z^* \in \operatorname{dom}\partial g^* \subseteq \operatorname{dom} g^*$. Furthermore, evaluating the last inequality in $y^* = z^*$, we deduce that $\liminf_i g^*(z^*_{\varphi(i)}) = g^*(z^*)$, and so there exists a second subnet $(z^*_{\psi(j)})_{j\in J}$ such that $g^*(z^*_{\psi(j)}) \to g^*(z^*)$. Since $(z^*_\alpha)$ was an arbitrary net in $D$ converging to $z^*$, we have that

$$\lim_{D\ni y^*\to z^*} g^*(y^*) = g^*(z^*).$$

The conclusion follows directly from Lemma 1.3.10. □

We can now state and prove our integration theorem.

**Theorem 1.3.12** *Let $f : X \to \mathbb{R}_\infty$ be an SDPD function and $g : X \to \mathbb{R}_\infty$ be any function satisfying the following condition:*

$$\partial f(x) \subseteq \partial g(x), \ \forall x \in X.$$

*Then there exists a constant $c \in \mathbb{R}$ such that*

$$\overline{\mathrm{co}}\, f = \overline{(\overline{\mathrm{co}}\, g)\Box\sigma_{\operatorname{dom} f^*}} + c.$$

*If, in addition, $\operatorname{dom} g^* \subseteq \overline{\operatorname{dom} f^*}$, then $\overline{\mathrm{co}}\, f$ and $\overline{\mathrm{co}}\, g$ are equal up to an additive constant.*

*Proof.* In this proof, the topological notation will refer to the Mackey topology $\tau(X^*, X^{**})$ for $X^*$. Without loss of generality, we may assume $g \in \Gamma_0(X, \theta)$ (due to the trivial inclusion $\partial g \subseteq \partial(\overline{\mathrm{co}}\, g)$). Because of the inclusion $\partial f(X) \subseteq \mathrm{dom}\, f^*$, it is clear that

$$\partial f(x) \subseteq \partial g(x) \cap \mathrm{dom}\, f^*, \ \forall x \in X.$$

Applying Proposition 1.2.22, we have that for all $x \in X$ with $\partial f(x) \neq \emptyset$,

$$\partial f(x) \subseteq \partial g(x) \cap \partial \sigma_{\mathrm{dom}\, f^*}(0) = \partial(g \Box \sigma_{\mathrm{dom}\, f^*})(x).$$

Defining $h = g \Box \sigma_{\mathrm{dom}\, f^*}$, the inclusion $\partial f(x) \subseteq \partial h(x)$ holds for all $x \in X$, and so $\mathrm{Cont}[f^*] \subseteq \mathrm{Cont}[h^*]$ (see Proposition 1.3.11). On the other hand, since $\mathrm{int}(\mathrm{dom}\, f^*) = \mathrm{int}\left(\overline{\mathrm{dom}\, f^*}\right)$, the equality $h^* = g^* + I_{\overline{\mathrm{dom}\, f^*}}$ provides the inclusion

$$\mathrm{Cont}[h^*] = \mathrm{Cont}[g^*] \cap \mathrm{Cont}[f^*] \subseteq \mathrm{Cont}[f^*],$$

which implies that $\mathrm{Cont}[f^*] = \mathrm{Cont}[h^*]$. Let $D$ be the dense subset of $\mathrm{Cont}[f^*]$ given in the definition of SDPD functions. We have that for all $x^* \in D$,

$$\partial f^*(x^*) = \overline{\mathrm{co}}^{w^{**}} \left[(\partial f)^{-1}(x^*)\right] \subseteq \overline{\mathrm{co}}^{w^{**}} \left[(\partial h)^{-1}(x^*)\right] \subseteq \partial h^*(x^*),$$

and so, applying Lemma 1.3.9, we have that $\partial f^*(x^*) \subseteq \partial h^*(x^*)$, for all $x^* \in \mathrm{Cont}[f^*]$. Now, all the hypotheses of Lemma 1.3.8 hold and so there exists a constant $c \in \mathbb{R}$ such that $f^* = h^* - c$. Finally,

$$\overline{\mathrm{co}}\, f = f^{**}\big|_X = h^{**}\big|_X + c = \overline{g \Box \sigma_{\mathrm{dom}\, f^*}} + c,$$

which finishes the first part of the proof. For the second part, note that if $\mathrm{dom}\, g^* \subseteq \overline{\mathrm{dom}\, f^*}$, then

$$h^* = g^* + I_{\overline{\mathrm{dom}\, f^*}} = g^*,$$

and so $h^{**}\big|_X = g^{**}\big|_X = g$, according to the fact that $g^*$ is proper. The conclusion is direct. □

### 1.3.3 Examples and SDPD spaces

This section gives some examples of certain classes of functions that are SDPD, and shows that the integrability property has a component that can be written as a property from the space where these functions are defined.

The first class involves the spaces with the Radon-Nikodým property (Radon-Nikodým spaces). Recall that a Banach space $X$ is known to have the Radon-Nikodým property (RNP, for short) if and only if each norm-epi-pointed function $f : X \to \mathbb{R}_\infty$ satisfies that its conjugate $f^* : X^* \to \mathbb{R}_\infty$ is Fréchet-differentiable in a dense set of $\mathrm{Cont}[f^*]$.

**Proposition 1.3.13** *Let $X$ be a Banach space with the RNP. Then every norm-epi-pointed and norm-lsc function on $X$ is SDPD.*

*Proof.* Let $f : X \to \overline{\mathbb{R}}$ be a norm-epi-pointed and norm-lsc function. In [17, Proposition 6], it is proved that if $f^*$ is Fréchet-differentiable at $x^* \in X^*$, then

$$\partial f^*(x^*) = (\partial f)^{-1}(x^*).$$

In particular, $f$ satisfies the equation (1.13) at each functional $x^* \in X^*$ where $f^*$ is Fréchet-differentiable. Thus $f$ is SDPD, since the set of point of Fréchet-differentiability of $f^*$ is a dense subset of $\operatorname{int}(\operatorname{dom} f^*)$ and that the Mackey topology $\tau(X^*, X^{**})$ on $X^*$ coincides with the norm topology on $X^*$. □

Recall that, for a nonempty set $K \subseteq X$, $\operatorname{ext}[K]$ denotes the set of extreme points of $K$, and for two points $x, y \in X$, $[x, y]$ denotes the convex segment between $x$ and $y$, that is,

$$[x, y] = \{tx + (1-t)y \ : \ t \in [0, 1]\}.$$

Recall also that a locally convex space $X$ is semi-reflexive if $(X^*, \beta(X^*, X))^* = X$, or equivalently, if the topologies $\tau(X^*, X)$ and $\beta(X^*, X)$ coincide in $X^*$.

We will show in Proposition 1.3.15 below that, in semi-reflexive spaces, $w$-lower semicontinuity is a sufficient condition to get SDPD function. To do so, we will need the following known lemma (which can be found in [5, Lemma 2.7.1]):

**Lemma 1.3.14** *Let $(X, \theta)$ be a locally convex space and let $C$ be a nonempty convex set of $X$. Let $x^* \in X^*$ be such that $\sigma_C(x^*) \in \mathbb{R}$ and consider the hyperplane*

$$H = \{x \in X \ : \ \langle x^*, x\rangle = \sigma_C(x^*)\}.$$

*Then, the set of extreme points* $\operatorname{ext}(C \cap H)$ *is included in* $\operatorname{ext}(C)$.

*Proof.* Let $x \in \operatorname{ext}(C \cap H)$ and suppose that there exists two points $x_1, x_2 \in C$ such that $x = \frac{1}{2}(x_1 + x_2)$. We have that since $x \in H$, we have that

$$\sigma_C(x^*) = \langle x^*, x\rangle = \tfrac{1}{2}\langle x^*, x_1\rangle + \tfrac{1}{2}\langle x^*, x_2\rangle,$$

and so, since $\langle x^*, x_1\rangle \le \sigma_C(x^*)$ and $\langle x^*, x_2\rangle \le \sigma_C(x^*)$, necessarily $\langle x^*, x_1\rangle = \langle x^*, x_2\rangle = \sigma_C(x^*)$.

Thus, $x_1, x_2 \in C \cap H$ and since $x \in \operatorname{ext}(C \cap H)$ we conclude that $x_1 = x_2 = x$. This entails that $x \in \operatorname{ext}(C)$, which proves the desire inclusion. □

**Proposition 1.3.15** *Let $X$ be a semi-reflexive locally convex space. Then every $\tau(X^*, X)$-epi-pointed and $w$-lower semicontinuous function on $X$ is SDPD.*

*Proof.* In this proof, the topological notation will refer to the $w$-topology for $X$ and the Mackey topology due to the primal, $\tau(X^*, X)$, for $X^*$. Because $X$ is semi-reflexive, $\tau(X^*, X)$ is the same as the Mackey topology over $X^*$ due to the bidual. Also, at each functional $x^* \in \mathrm{Cont}[f^*]$, the subdifferential $\partial f^*(x^*)$ is a nonempty convex $w$-compact subset of $X$. Thus, by the Krein-Milman theorem (see Theorem 1.1.4) for each $x^* \in \mathrm{Cont}[f^*]$,

$$\partial f^*(x^*) = \overline{\mathrm{co}}\left[\mathrm{ext}[\partial f^*(x^*)]\right].$$

Therefore, it is sufficient to prove that for any functional $x^* \in \mathrm{Cont}[f^*]$,

$$\mathrm{ext}(\partial f^*(x^*)) \subseteq (\partial f)^{-1}(x^*). \tag{1.14}$$

Let us consider then a functional $x^* \in \mathrm{Cont}[f^*]$ and the function $\sigma_{\mathrm{epi}\, f}$ $( = (I_{\overline{\mathrm{co}}(\mathrm{epi}\, f)})^*)$ on $X^* \times \mathbb{R}$ (endowed with the product topology $\tau(X^*, X) \times \tau_0$). For $\alpha > 0$ we have that

$$\begin{aligned}\sigma_{\mathrm{epi}\, f}(x^*, -\alpha) &= \sup\{\langle x^*, x\rangle - \alpha f(x) \mid x \in \mathrm{dom}\, f\}\\ &= \alpha \sup\left\{\langle \alpha^{-1} x^*, x\rangle - f(x) \mid x \in \mathrm{dom}\, f\right\}\\ &= \alpha f^*(\alpha^{-1} x^*).\end{aligned}$$

Then, because $f^*$ is continuous at $x^*$, $\sigma_{\mathrm{epi}\, f}$ is continuous at $(x^*, -1)$. In particular, for each $\varepsilon > 0$, $\partial_\varepsilon \sigma_{\mathrm{epi}\, f}(x^*, -1)$ is a nonempty, convex and $w \times \tau_0$-compact set. Let us now fix an $\varepsilon > 0$ and define the set

$$C := \partial_\varepsilon \sigma_{\mathrm{epi}\, f}(x^*, -1).$$

Because $\partial \sigma_{\mathrm{epi}\, f}(x^*, -1) \subseteq C$ we have that $\partial \sigma_C(x^*, -1) \equiv \partial \sigma_{\mathrm{epi}\, f}(x^*, -1)$. Also,

$$\begin{aligned}C &= \{(x, \lambda) \in X \times \mathbb{R} \mid \langle x^*, x\rangle - \lambda \geq \sigma_{\mathrm{epi}\, f}(x^*, -1) + I_{\overline{\mathrm{co}}(\mathrm{epi}\, f)}(x, \lambda) - \varepsilon\}\\ &= \overline{\mathrm{co}}(\mathrm{epi}\, f) \cap \{(x, \lambda) \in X \times \mathbb{R} \mid \langle x^*, x\rangle - \lambda \geq \sigma_{\mathrm{epi}\, f}(x^*, -1) - \varepsilon\}.\end{aligned}$$

Since $\mathrm{ext}[\overline{\mathrm{co}}(\mathrm{epi}\, f)] \cap C \subseteq C$, by Lemma 1.3.14, we have that

$$\mathrm{ext}\left[\partial \sigma_{\mathrm{epi}\, f}(x^*, -1)\right] \subseteq \mathrm{ext}[\overline{\mathrm{co}}(\mathrm{epi}\, f)] \cap C \subseteq \mathrm{ext}(C).$$

Consider now the set

$$H = \overline{\mathrm{co}}(\mathrm{epi}\, f) \cap \{(x, \lambda) \in X \times \mathbb{R} \mid \langle x^*, x\rangle - \lambda = \sigma_{\mathrm{epi}\, f}(x^*, -1) - \varepsilon\}.$$

It is clear that $\overline{\mathrm{co}}(H \cup [C \cap \mathrm{epi}\, f]) \subseteq C$. For the reverse inclusion, let us consider $(x, \lambda) \in C$. Because $C \subseteq \overline{\mathrm{co}}(\mathrm{epi}\, f)$, there exists a net $(x_i, \lambda_i)_{i \in \Lambda}$ in $\mathrm{co}(\mathrm{epi}\, f)$ converging to $(x, \lambda)$. We have two cases: First, $\langle x^*, x\rangle - \lambda = \sigma_{\mathrm{epi}\, f}(x^*, -1) - \varepsilon$ which means that $(x, \lambda) \in H \subseteq \overline{\mathrm{co}}(H \cup [C \cap \mathrm{epi}\, f])$. Second, $\langle x^*, x\rangle - \lambda > \sigma_{\mathrm{epi}\, f}(x^*, -1) - \varepsilon$. In such a case, there exists $i_0 \in \Lambda$, such that,

$$\langle x^*, x_i\rangle - \lambda_i > \sigma_{\mathrm{epi}\, f}(x^*, -1) - \varepsilon,\ \forall i \geq i_0.$$

In particular, $(x_i, \lambda_i)_{i \geq i_0}$ is contained in $C \cap \mathrm{co}(\mathrm{epi}\, f)$. Note that, for each $i \geq i_0$, we can find $(x_i^1, \lambda_i^1), (x_i^2, \lambda_i^2) \in \mathrm{co}(\mathrm{epi}\, f)$ and $t_i \in [0, 1]$ such that

- $(x_i^1, \lambda_i^1) \in \mathrm{co}(C \cap \mathrm{epi}\, f)$, which implies that $\langle x^*, x_i^1 \rangle - \lambda_i^1 \geq \sigma_{\mathrm{epi}\, f}(x_i^1, \lambda_i^1) - \varepsilon$.
- $(x_i^2, \lambda_i^2) \in \mathrm{co}(\mathrm{epi}\, f \setminus C)$, which implies that $\langle x^*, x_i^2 \rangle - \lambda_i^2 \leq \sigma_{\mathrm{epi}\, f}(x_i^2, \lambda_i^2) - \varepsilon$.
- $(x_i, \lambda_i) = t_i(x_i^1, \lambda_i^1) + (1 - t_i)(x_i^2, \lambda_i^2)$.

Then, it is direct from the first two conditions that we can choose a point $(x_i^3, \lambda_i^3) \in [(x_i^1, \lambda_i^1), (x_i^2, \lambda_i^2)] \cap H$, and because $(x_i, \lambda_i) \in C$ for all $i \geq i_0$, we know that

$$(x_i, \lambda_i) \in [(x_i^1, \lambda_i^1), (x_i^3, \lambda_i^3)] \subseteq \overline{\mathrm{co}}(H \cup [C \cap \mathrm{epi}\, f]).$$

Finally, $(x, \lambda) \in \overline{\mathrm{co}}(H \cup [C \cap \mathrm{epi}\, f])$, concluding that $\overline{\mathrm{co}}(H \cup [C \cap \mathrm{epi}\, f]) = C$. Applying the Milman theorem (see [23, Theorem 3.66]), provided that $H$ and $C \cap \mathrm{epi}\, f$ are $w \times \tau_0$-closed (and thus their union) and the trivial fact that $\partial \sigma_{\mathrm{epi}\, f}(x^*, -1) \cap H = \emptyset$, we have that

$$\mathrm{ext}\, [\partial \sigma_{\mathrm{epi}\, f}(x^*, -1)] \subseteq \mathrm{epi}\, f. \tag{1.15}$$

To conclude, let us recall three known facts:

(i) $(x, \lambda) \in \partial \sigma_{\mathrm{epi}\, f}(x^*, -1) \implies \lambda = \overline{\mathrm{co}}\, f(x)$.

(ii) $x \in \partial f^*(x^*)$ if and only if $(x, \overline{\mathrm{co}}\, f(x)) \in \partial \sigma_{\mathrm{epi}\, f}(x^*, -1)$.

(iii) $x \in (\partial f)^{-1}(x^*)$ if and only if $x \in \partial f^*(x^*)$ and $f(x) = \overline{\mathrm{co}}\, f(x)$.

Take now $x \in \mathrm{ext}(\partial f^*(x^*))$. We have that $(x, \overline{\mathrm{co}}\, f(x)) \in \mathrm{ext}[\partial \sigma_{\mathrm{epi}\, f}(x^*, -1)]$. If not, due to (i), there would be two distinct points $(x_1, \overline{\mathrm{co}}\, f(x_1)), (x_2, \overline{\mathrm{co}}\, f(x_2))$ in $\partial \sigma_{\mathrm{epi}\, f}(x^*, -1)$ and a real number $t \in (0, 1)$ such that

$$(x, \overline{\mathrm{co}}\, f(x)) = t(x_1, \overline{\mathrm{co}}\, f(x_1)) + (1 - t)(x_2, \overline{\mathrm{co}}\, f(x_2)).$$

In particular, $x_1 \neq x_2$ and $x = tx_1 + (1-t)x_2$, and because of (ii), $x_1, x_2 \in \partial f^*(x^*)$, which is a contradiction. Then, applying the inclusion (1.15), we have that $(x, \overline{\mathrm{co}}\, f(x)) \in \mathrm{epi}\, f$ and so $\overline{\mathrm{co}}\, f(x) = f(x)$. Finally, because of (iii), $x \in (\partial f)^{-1}(x^*)$ , which proves the inclusion (1.14), finishing the proof. □

What do the semi-reflexive spaces and the normed spaces with the Radon-Nikodým property have in common that allow the previous functions to be SDPD? It is not easy to give an answer, but clearly it is related to the subdifferential of the conjugate functions. The following small characterization helps us to understand this situation better:

**Proposition 1.3.16** *For a function $f : X \to \overline{\mathbb{R}}$, a functional $x^* \in X^*$ satisfies the equality (1.13) if and only if the following two conditions hold:*

*(i)* $X \cap \partial f^*(x^*) = \overline{\mathrm{co}}\, [(\partial f)^{-1}(x^*)]$.

*(ii)* $\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}}$.

*In particular, a* $\tau(X^*, X^{**})$*-epi-pointed function is SDPD whenever there exists a* $\tau(X^*, X^{**})$*-dense subset* $D$ *of* $\mathrm{Cont}[f^*, \tau(X^*, X^{**})]$ *such that every* $x^* \in D$ *satisfies (i) and (ii).*

*Proof.* The second part of the proposition is direct, so we will show only the first equivalence:

$\Leftarrow$) Assuming (i) and (ii), it is direct that

$$\begin{aligned}\overline{\mathrm{co}}^{w^{**}}[(\partial f)^{-1}(x^*)] &= \overline{\overline{\mathrm{co}}[(\partial f)^{-1}(x^*)]}^{w^{**}}\\ &\overset{(i)}{=} \overline{X \cap \partial f^*(x^*)}^{w^{**}}\\ &\overset{(ii)}{=} \partial f^*(x^*).\end{aligned}$$

$\Rightarrow$) Suppose now that $x^* \in X^*$ satisfies (1.13). Then

$$\begin{aligned}X \cap \partial f^*(x^*) &= X \cap \overline{\mathrm{co}}^{w^{**}}[(\partial f)^{-1}(x^*)]\\ &= \overline{\mathrm{co}}^{w}[(\partial f)^{-1}(x^*)]\\ &= \overline{\mathrm{co}}[(\partial f)^{-1}(x^*)],\end{aligned}$$

so (i) holds. Finally, noting again that

$$\overline{\mathrm{co}}^{w^{**}}[(\partial f)^{-1}(x^*)] = \overline{\overline{\mathrm{co}}[(\partial f)^{-1}(x^*)]}^{w^{**}},$$

the statement (ii) is concluded, according to (1.13) and to (i).

□

So, the necessary condition that doesn't depend on the function $f$ but entirely on the conjugate is that the equation

$$\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}} \tag{1.16}$$

must hold in a $\tau(X^*, X^{**})$-dense set of functionals. Motivated by this observation, we introduce the following definition:

**Definition 1.3.17** (SDPD space) *We say that a locally convex space* $X$ *is an SDPD space if for every function* $f^* \in \Gamma_0(X^*, w^*)$ *with* $\mathrm{Cont}[f^*, \tau(X^*, X^{**})]$ *nonempty, there exists a* $\tau(X^*, X^{**})$*-dense subset* $D$ *of* $\mathrm{Cont}[f^*, \tau(X^*, X^{**})]$*, such that for each* $x^* \in D$*, the equation* (1.16) *holds.*

Note that in an SDPD space, each $\tau(X^*, X^{**})$-epi-pointed function in $\Gamma_0(X)$ is an SDPD function, since for each $f \in \Gamma_0(X)$ we have that

$$X \cap \partial f^*(x^*) = (\partial f)^{-1}(x^*), \ \forall x^* \in X^*,$$

and so the condition (i) in Proposition 1.3.16 always holds. In fact, this is a characterization: The locally convex space $X$ is an SDPD space if and only if each $\tau(X^*, X^{**})$-epi-pointed function in $\Gamma_0(X)$ is an SDPD function.

Now, we establish a second characterization of SDPD spaces as an RNP-like property:

**Proposition 1.3.18** *Let $X$ be a locally convex space, $f \in \Gamma_0(X)$ be a $\tau(X^*, X^{**})$-epi-pointed function and $x^* \in \operatorname{Cont}[f^*, \tau(X^*, X^{**})]$. We have that*

$$\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}} \iff (f^*)'(x^*, \cdot) \textit{ is } w^*\textit{-lsc.}$$

*In particular, $X$ is an SDPD space if and only if for each function $f \in \Gamma_0(X)$, the set*

$$D = \{x^* \in \operatorname{int}(\operatorname{dom} f^*, \tau(X^*, X^{**})) \ : \ (f^*)'(x^*, \cdot) \textit{ is } w^*\textit{-lsc.}\}$$

*is $\tau(X^*, X^{**})$-dense in* $\operatorname{dom} f^*$.

*Proof.* The second part of the proposition is direct from the first part, so we only need to prove that one. Assume then that $f \in \Gamma(X)$ and $x^* \in \operatorname{Cont}[f^*, \tau(X^*, X^{**})]$. On one hand, we know (see comments before equation (1.11)) that

$$(f^*)'(x^*, u^*) = \sup\{\langle x^{**}, u^*\rangle \ : \ x^{**} \in \partial f^*(x^*)\}, \ \forall u^* \in X^*.$$

On the other hand, Considering the duality $\langle X, X^*\rangle$ and using Proposition 1.2.27, we have that

$$\overline{(f^*)'(x^*, \cdot)}^{w^*}(u^*) = \sup\{\langle x, u^*\rangle \ : \ x \in X \cap \partial f^*(x^*)\}, \ \forall u^* \in X^*.$$

Therefore,

$$\begin{aligned} (f^*)'(x^*, \cdot) = \overline{(f^*)'(x^*, \cdot)}^{w^*} &\iff \sigma_{\partial f^*(x^*)} = \sigma_{X \cap \partial f^*(x^*)} \\ &\iff \overline{\operatorname{co}}^{w^{**}}[\partial f^*(x^*)] = \overline{\operatorname{co}}^{w^{**}}[X \cap \partial f^*(x^*)] \\ &\iff \partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}}, \end{aligned}$$

which finishes the proof. □

From the definition of SDPD spaces follows this direct corollary:

**Corollary 1.3.19** *Suppose that $X$ is an SDPD space. Let $f : X \to \mathbb{R}_\infty$ be a $\tau(X^*, X^{**})$-epi-pointed function and define the sets*

$$\begin{aligned} D_1 &= \left\{x \in \operatorname{Cont}[f^*, \tau(X^*, X^{**})] \mid \partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}}\right\}. \\ D_2 &= \left\{x \in \operatorname{Cont}[f^*, \tau(X^*, X^{**})] \mid X \cap \partial f^*(x^*) = \overline{\operatorname{co}}\left[(\partial f)^{-1}(x^*)\right]\right\}. \end{aligned}$$

*Then $f$ is SDPD if and only if $D_2 \cap D_1$ is $\tau(X^*, X^{**})$-dense in $D_1$.*

# Chapter 2

# Convex smooth-like and $w^*$-smooth-like properties in Banach spaces

The aim of this chapter and the following one is to study SDPD spaces in the Banach space setting. As we have seen in Proposition 1.3.18, SDPD property can be regarded as a sort of "smoothness property". This chapter is devoted to formalize this idea introducing the notion of *convex smooth-like properties* and *convex $w^*$-smooth-like properties.*

The development of those concepts have been done under the light of Asplund spaces theory, widely developed in the literature. For that reason, we give a quick summary of Fréchet-differentiability of convex functions and Asplund spaces in Sections 2.1 and 2.2, respectively. We will lead the reader to Theorems 2.2.13 and 2.2.15, which fully established one the most important results in this theory: The duality with the Radon-Nikodým Property.

Our contribution is contained in Section 2.3, and it will be fundamental to follow what is developed in Chapter 3.

## Notation

In the following, $X$ will be a real Banach space, with dual $X^*$ and bidual $X^{**}$. We will write $\|\cdot\|$ to denote the norm of $X$, $X^*$ and $X^{**}$ indifferently, and $\tau_{\|\cdot\|}$ to denote the topology induced by it. Also, we will write $w$, $w^*$ and $w^{**}$ to denote the weak-topology on $X$ and on $X^*$, the weak-star topology on $X^*$, and the weak-star topology on $X^{**}$ (given by $X^*$), respectively.

Recalling that for any normed space $X$, $\tau_{\|\cdot\|}$ coincides with the Mackey topology $\tau(X, X^*)$, the notation $\Gamma_0(X)$ will always stand for $\Gamma_0(X, \|\cdot\|)$ (see equation (1.5) and Remark 1.2.5).

Whenever we consider an equivalent norm $p$ on $X$, it will be useful to denote by $p_*$ the associated dual norm on $X^*$, that is

$$p_*(x^*) := \sup\{\langle x^*, x\rangle \ : \ x \in X \text{ with } p(x) \leq 1\}. \tag{2.1}$$

We will also write $\|\cdot\|_*$ instead of $\|\cdot\|$, whenever it is convenient. Also, we will write $\mathbb{B}_{(X,p)}$ and $\mathbb{S}_{(X,p)}$ to denote the unit ball and the unit sphere with respect to $p$, respectively (and the analogous notation in $X^*$ for the norm $p_*$). If the norm is not specified, we will assume that we are using the initial norm $\|\cdot\|$.

Recall that an equivalent norm $q$ on the dual space $X^*$ is a dual norm (that is, $q = p_*$ for some equivalent norm $p$ on $X$) if and only if it is $w^*$-lower semicontinuous.

In order to reduce notation, for any two Banach spaces $X$ and $Y$, any subset $U$ of $X$ and any mapping $\varphi : U \subseteq X \to Y$, we will say that $\varphi$ is F-differentiable at a point $u \in U$ if it is Fréchet-differentiable at $u$. Analogously, we will say that $\varphi$ is G-differentiable at a point $u$ if it is Gâteaux-differentiable at $u$.

Also, we will consider the convention that $\mathbb{N}$ is the set of all positive integers, that is, $0 \notin \mathbb{N}$.

The rest of the notation is either classic (see, e.g., [16],[5] or [36]) or has been already presented in the Introduction of Part I and in Chapter 1.

## 2.1 Differentiability of Convex Functions and Support functionals

In view of our study of smooth-like properties of convex functions in Section 2.3, we will recall diverse results related to some useful properties, in the thesis, of convex functions. The present section is devoted to differentiability properties of convex functions defined over a Banach space. We first state the following lemma (see, e.g., [36, Proposition 3.3]).

**Lemma 2.1.1** *Let $X$ be a Banach space and $f \in \Gamma_0(X)$. Then,* $\operatorname{int}(\operatorname{dom} f) = \operatorname{Cont}[f]$.

By this lemma, we know that for every function $f \in \Gamma_0(X)$ and every point $x \in \operatorname{int}(\operatorname{dom} f)$, we have that $f'(x;\cdot)$ is real-valued and continuous. Thus, if $f'(x;\cdot)$ is also linear, then by definition we have that $f$ is G-differentiable at $x$ with $\nabla f(x) = f'(x;\cdot)$. Furthermore, applying Proposition 1.2.27 we can state the following proposition:

**Proposition 2.1.2** *Let $f \in \Gamma_0(X)$ and $x \in \operatorname{int}(\operatorname{dom} f)$. The following assertions are equivalent:*

*(a) $f'(x;\cdot)$ is linear.*

*(b) $f$ is G-differentiable at $x$.*

*(c) The subdifferential $\partial f(x)$ is a singleton.*

*(d) There exists a unique functional $x^* \in X^*$ satisfying $\langle x^*, h\rangle \leq f'(x;h)$, for all $h \in X$.*

*In such a case, $\partial f(x) = \{\nabla f(x)\}$.*

It is known (see, e.g., [36, Proposition 1.23 and Exercice 1.24]) that the G-differentiability and F-differentiability of a function $f \in \Gamma_0(X)$ at a point $x \in \operatorname{int}(\operatorname{dom} f)$ can be characterized without using the gradient $\nabla f(x)$, as follows.

**Proposition 2.1.3** *Let $f \in \Gamma_0(X)$ and $x \in \operatorname{int}(\operatorname{dom} f)$. Then,*

*(a) $f$ is G-differentiable at $x$ if and only if for every $h \in X$,*

$$\lim_{t \searrow 0} \frac{f(x+th) + f(x-th) - 2f(x)}{t} = 0.$$

*(b) $f$ is F-differentiable at $x$ if and only if $\delta > 0$ such that*

$$\lim_{h \to 0} \frac{f(x+th) + f(x-th) - 2f(x)}{\|h\|} = 0. \tag{2.2}$$

From this characterization, it is possible to show that the set of points where a function $f \in \Gamma_0(X)$ is a $G_\delta$-subset of $\operatorname{int}(\operatorname{dom} f)$ (see, e.g., [36, Proposition 1.25]).

**Corollary 2.1.4** *Let $f \in \Gamma_0(X)$. The set $G = \{x \in \operatorname{dom} f \ : \ f$ is F-differentiable at $x\}$ is a $G_\delta$ (possible empty) subset of* $\operatorname{int}(\operatorname{dom} f)$.

The following theorem (see, e.g., [36, Ch. 2]) characterizes the F-differentiability in terms of the subdifferential of the function.

**Theorem 2.1.5** (Asplund-Rockafellar) *Let $f \in \Gamma_0(X)$ and $x \in \operatorname{int}(\operatorname{dom} f)$. We have that $f$ is F-differentiable at $x$ if and only if $\partial f : X \rightrightarrows X^*$ is single-valued and $\tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$-usc at $x$.*

Concerning conjugate functions in the Banach space setting, the subdifferential has a very interesting behavior related to support functionals. In the following sections as well as in chapter 3, we will use this properties, so we will present them now.

Recall that for a nonempty closed convex set $K$, a nonzero functional $x^* \in X^*$ is called *support functional of $K$* if $x^*$ attains its supremum in $K$ at some point, that is, if there exists $x \in K$ such that $\langle x^*, x\rangle = \sigma_K(x^*)$, where $\sigma_K$ is the support function of the set $K$ (see Example 1.2.2). In such a case, $x$ is called a *support point of $K$*, since it is supported by $x^*$. We will denote by $S(K)$ the set of support functionals of $K$.

We will show that the set of support functionals of a convex closed set $K$ is dense in the cone of linear functionals that are bounded above on $K$. To do so, we will need a classic theorem called the Brønsted-Rocafellar variational principle [11]. It can be derived from the fundamental Ekeland variational principle and such a proof can be found, for example, in [36, Theorem 3.17].

**Theorem 2.1.6** (Brønsted-Rockafellar) *Let $f \in \Gamma_0(X)$. Then, given any point $x_0 \in \operatorname{dom} f$, any two constants $\varepsilon, \lambda > 0$ and any linear functional $x_0^* \in \partial_\varepsilon f(x_0)$, there exists $x \in \operatorname{dom} f$ and $x^* \in X^*$ such that*

*(a)* $x^* \in \partial f(x)$.

*(b)* $\|x - x_0\| \leq \varepsilon/\lambda$.

*(c)* $\|x^* - x_0^*\| \leq \lambda$.

*In particular,* $\operatorname{dom} \partial f$ *is dense in* $\operatorname{dom} f$.

Noting that $x^*$ is a support functional of $K$ if and only if $X \cap \partial\sigma_K(x^*) \neq \emptyset$, we can write

$$S(K) = \{x^* \in \operatorname{dom} \sigma_K \ : \ X \cap \partial\sigma_K(x^*) \neq \emptyset\}. \tag{2.3}$$

Therefore, this notion can be extended to all functions in $\Gamma_0(X^*, w^*)$. In what follows, we will say that a linear functional $x^* \in X^*$ *supports* a function $f \in \Gamma_0(X)$ if $X \cap \partial f^*(x^*) \neq \emptyset$, and we will denote the set of support functionals of $f$ as $S(f)$.

**Proposition 2.1.7** *Let $f \in \Gamma_0(X)$. Then the set of support functionals of $f$*

$$S(f) = \{x^* \in \operatorname{dom} f^* \ : \ X \cap \partial f^*(x) \neq \emptyset\}$$

*is dense in* $\operatorname{dom} f^*$.

*Proof.* Fix $f \in \Gamma_0(X)$, $x_0^* \in \operatorname{dom} f^*$ and $\varepsilon > 0$. Since $\partial_\varepsilon f^*(x_0^*) = \overline{X \cap \partial_\varepsilon f^*(x_0^*)}^{w^{**}}$ and $\partial_\varepsilon f^*(x_0^*) \neq \emptyset$, we can choose a point $x_0 \in X \cap \partial_\varepsilon f^*(x_0)$. Using Proposition 1.2.24 (parts $(b)$ and $(d)$), we get that $x_0 \in \operatorname{dom} f$ and $x_0^* \in \partial_\varepsilon f(x_0)$.

Applying Theorem 2.1.6 with $\lambda = \varepsilon$, we get that there exists $(x, x^*) \in \partial f$ such that $\|x - x_0\| \leq 1$ and $\|x^* - x_0^*\| \leq \varepsilon$. In particular, using Proposition 1.2.18, we get that $x \in \partial f^*(x^*)$, and so $x^* \in S(f)$. Thus, $B_{X^*}(x_0^*, \varepsilon) \cap S(f) \neq \emptyset$, which proves the density by arbitrariness of $\varepsilon$. $\square$

**Theorem 2.1.8** (Bishop-Phelps) *Let $K$ be a nonempty closed convex set of a Banach space $X$. Then,*

*(a) The set of support points of $K$ is dense in* $\operatorname{bd} K$.

(b) *The set of support functionals of $K$, denoted by $S(K)$, is dense in the cone of all those functionals which are bounded above on $K$.*

*Proof.* Part ($b$) follows directly from Proposition 2.1.7 applied to $f = I_K$. For part ($a$), suppose that $x_0 \in \operatorname{bd} K$ and fix $\varepsilon \in ]0,1[$. Choose $x_1 \in X \setminus K$ with $\|x_1 - x_0\| \leq \varepsilon$. Using the Hahn-Banach separation theorem, there exists $x_0^* \in \mathbb{S}_{X^*}$ such that $\sigma_K(x_0^*) < \langle x_0^*, x_1\rangle$. Thus, for all $x \in K$ we can write

$$\langle x_0^*, x\rangle < \langle x_0^*, x_1\rangle \leq \langle x_0^*, x_1 - x_0\rangle + \langle x_0^*, x_0\rangle \leq \langle x_0^*, x_0\rangle + \varepsilon.$$

Therefore, $\langle x_0^*, x - x_0\rangle \leq I_K(x) - I_K(x_0) + \varepsilon$, which entails that $x^* \in \partial_\varepsilon I_K(x_0)$. Applying Theorem 2.1.6 with $f = I_K$ and $\lambda = \sqrt{\varepsilon}$, there exists $x \in \operatorname{dom} f = K$ and $x^* \in \partial f(x)$ such that

$$\|x_0 - x\| \leq \sqrt{\varepsilon} \qquad \text{and} \qquad \|x^* - x_0^*\| \leq \sqrt{\varepsilon} < 1.$$

Since $x_0^* \in \mathbb{S}_{X^*}$, this yields $\|x^*\| > 0$, and so $x \in \operatorname{bd} K$, since for every $y \in \operatorname{int}(K)$, we easily verify that $\partial f(y) = \{0\}$. Since $\varepsilon > 0$ is arbitrary, the proof is complete. $\square$

Finally, for a function $f \in \Gamma_0(X)$ is such that its conjugate $f^*$ is F-differentiable at some point $x^* \in \operatorname{dom} f^*$, then the differentiability point $x^*$ is a support functional of the function $f$, as the following proposition states. This proposition can be found in [24, Lemma 3] and also in [13, Corollary 3.3.4].

**Proposition 2.1.9** *Let $X$ be a Banach space and $f \in \Gamma_0(X)$. If the conjugate function $f^*$ is F-differentiable at a point $x^* \in \operatorname{dom} f^*$, then $\nabla f^*(x^*) \in X$.*

## 2.2 Asplund spaces and Radon-Nikodým Property

In addition to the properties of differentiability fo convex functions recalled in the previous section, the present one is focused on some properties of Asplund (resp. Radon-Nikodým) spaces which will be use later. These properties are our main motivation for the theory developed in section 2.3.

### 2.2.1 Asplund Spaces

**Definition 2.2.1** (Asplund space) *A Banach space $X$ is said to be an* Asplund space *if for every function $f \in \Gamma_0(X)$, the set*

$$\mathfrak{D}(f) := \{x \in \operatorname{dom} f \ : \ f \text{ is F-differentiable at } x\} \tag{2.4}$$

*is dense in* $\operatorname{int}(\operatorname{dom} f)$.

In what follows, for a function $f \in \Gamma_0(X)$ we will keep the notation $\mathfrak{D}(f)$ to denote the set of points of $\operatorname{dom} f$ at which $f$ is F-differentiable. Observe that, according to Corollary 2.1.4, the set $\mathfrak{D}(f)$ is in fact a $G_\delta$-dense set of $\operatorname{int}(\operatorname{dom} f)$, whenever it is dense.

**Example 2.2.2** (Asplund spaces) Some classic examples in the literature (see, e.g., [9], [21] or [36]) are the followings:

(a) Every reflexive Banach space is an Asplund space.

(b) As stated in Theorem 2.2.6, a separable Banach space is an Asplund space if and only if its dual is separable. So, $c_0$ is an Asplund space, but $\ell^1$ isn't.

(c) A Banach space $X$ is said to be *weakly compactly generated* (*WCG*, for short) if there exists a $w$-compact subset $K$ of $X$ such that $X = \overline{\operatorname{span}}(K)$. It is known that whenever a Banach space has a WCG dual, then it is an Asplund space (see, e.g., [36, Theorem 2.43]).

(d) $L^1[0,1]$ is not an Asplund space.

□

As we will see later in the following section, Asplund property (that is, to be an Asplund space) has a dual *geometrical interpretation*. First, we introduce here a first glance of what this duality is about. To do so, we need to introduce the notion of slice.

**Definition 2.2.3** (Slices) *Let $X$ be a Banach space and $C$ be a nonempty subset of $X$. For a functional $x^* \in X^* \setminus \{0\}$ and a real $\alpha > 0$, we define the* slice of $C$ induced by $x^*$ and $\alpha$ *as the set*

$$S(C, x^*, \alpha) := \{x \in C \ : \ \langle x^*, x\rangle > \sigma_C(x^*) - \alpha\}. \tag{2.5}$$

*If $C$ is a subset of the dual space $X^*$ and $x \in X \setminus \{0\}$, the slice $S(C, x, \alpha)$ is called the* $w^*$-slice of $C$ induced by $x$ and $\alpha$, *to emphasize that it is given by a functional of the primal space.*

Clearly, the slices (resp. $w^*$-slices) are relatively open subsets of $C$ and they are nonempty whenever $\sigma_C(x^*) < +\infty$ (resp. $\sigma_C(x) < +\infty$). We also can define the notion of *closed slice of $C$ induced by $x^*$ and $\alpha$* as the set

$$\overline{S}(C, x^*, \alpha) := \{x \in X \ : \ \langle x^*, x\rangle \geq \sigma_C(x^*) - \alpha\}. \tag{2.6}$$

If $C$ is closed and convex in $X$, we easily see that

$$x \in \overline{S}(C, x^*, \alpha) \iff \langle x^*, x\rangle \geq \sigma_C(x^*) + I_C(x) - \alpha \iff x \in \partial_\alpha \sigma_C(x^*). \tag{2.7}$$

We also can define the closed $w^*$-slices, and derive the analogous to equation 2.7 when the set $C$ is $w^*$-closed and convex in $X^*$. Equation 2.7 and its dual form are often used in the text.

The next Theorem 2.2.5 involving the reductions of Asplund spaces to other families of functions is well-known. Since this theorem is the main motivation of our contribution in this chapter, we provide a proof of its content as well as the justification of the following lemma involved in the arguments therein. The proof of Lemma 2.2.4 can be found in [36, Lemma 2.18 and Theorem 2.30], while the proof of Theorem 2.2.5 is more disperse: It can be derived mixing up different propositions contained in the books [9], [21] and [36].

**Lemma 2.2.4** *A Banach space $X$ is an Asplund space if and only if for every nonempty bounded subset $K$ of $X^*$ and for every $\varepsilon > 0$, there exists a $w^*$-slice $S(K, x, \alpha)$ with diameter less than $\varepsilon$.*

*Proof.* Let us start with the necessity: Reasoning by absurd, suppose that there exists a nonempty bounded subset $K$ of $X^*$ and a real $\varepsilon > 0$ such that every $w^*$-slice of $K$ has diameter strictly larger than $\varepsilon$.

To arrive at a contradiction, we will show that $\sigma_K$, which is continuous and real-valued due to the boundedness of $K$, is nowhere Fréchet-differentiable. Since $K$ is not a singleton, it is clear that $\sigma_K$ is not F-differentiable at 0. Fix then $x \in X \setminus \{0\}$ and, for each $n \in \mathbb{N}$ choose $x_n^*, y_n^* \in S(K, x, \varepsilon/3n)$ such that $\|x_n^* - y_n^*\| > \varepsilon$. We then know, that there exists $x_n \in \mathbb{S}_X$ such that $\langle x_n^* - y_n^*, x_n\rangle > \varepsilon$.

Thus, we can write

$$\begin{aligned}\sigma_K(x + \tfrac{1}{n}x_n) + \sigma_K(x - \tfrac{1}{n}x_n) - 2\sigma_K(x) &\geq \langle x_n^*, x + \tfrac{1}{n}x_n\rangle + \langle y_n^*, x + \tfrac{1}{n}x_n\rangle - 2\sigma_K(x)\\ &\geq \langle x_n^* - y_n^*, \tfrac{1}{n}x_n\rangle - \tfrac{2\varepsilon}{3n}\\ &> \tfrac{\varepsilon}{n} - \tfrac{2\varepsilon}{3n} = \tfrac{\varepsilon}{3n},\end{aligned}$$

where the second inequality follows from the facts that $\sigma_K(x) \leq \langle x_n^*, x\rangle + \varepsilon/3n$ and $\sigma_K(x) \leq \langle y_n^*, x\rangle + \varepsilon/3n$, due to the inclusion $S(K, x, \varepsilon/3n) \subseteq \partial_{\varepsilon/3n}\sigma_K(x)$ for all $n \in \mathbb{N}$, and by Proposition 1.2.24(b). Dividing the latter inequality by $\frac{1}{n}$, we get that $\sigma_K$ cannot be F-differentiable at $x$, according to Proposition 2.1.3. The proof of the necessity is then finished, since $x$ is an arbitrary point of $X \setminus \{0\}$.

To prove the sufficiency, fix $f \in \Gamma_0(X)$ with $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$. By Propositions 1.2.20, we know that $W := \operatorname{int}(\operatorname{dom}\partial f)$ coincides with $\operatorname{int}(\operatorname{dom} f)$. We will show that there exists a $G_\delta$-dense subset $G$ of $W$ on which $\partial f$ is a singleton and it is norm-norm-usc.

For each $n \in \mathbb{N}$, define the set $G_n$ as all points $x \in W$ such that there exists a neighborhood $V \in \mathcal{N}_X(x)$ with $\operatorname{diam}(\partial f(V)) < \frac{1}{n}$. It is not hard to realize that the set $G = \bigcap_{n\in\mathbb{N}} G_n$ is the collection of all points $x \in W$ such that $\partial f(x)$ is a singleton and $\partial f$ is norm-norm-usc at $x$. Thus, to prove our claim we only need to show that $G_n$ is dense in $W$, since by construction is already open.

Fix $n \in \mathbb{N}$, $x_0 \in W$ and an open neighborhood $U \in \mathcal{N}_X(x_0)$ with $U \subseteq W$. By Proposition 1.2.21, we may assume that $\partial f(U)$ is bounded. By hypothesis, there exists $z \in X$ and

$\alpha > 0$ such that

$$\operatorname{diam}(S(\partial f(U), z, \alpha)) = \operatorname{diam}\{x^* \in \partial f(U) \ : \ \langle x^*, z\rangle > \sigma_{\partial f(U)}(z) - \alpha\} < \frac{1}{n}.$$

Let us denote $S := S(\partial f(U), z, \alpha)$. Since $S$ is nonempty, there exists $x_1 \in U$ and $x^* \in \partial f(x_1)$ such that $x^* \in S$. Fix $r > 0$ small enough such that $x_2 := x_1 + rz \in U$. We will show that $\partial f(x_2) \subseteq \{x^* \in X^* \ : \ \langle x^*, z\rangle > \sigma_{\partial f(U)}(z) - \alpha\} =: Q$. Indeed, fixing $y^* \in \partial f(x_2)$ we have by monotonicity that

$$0 \leq \langle y^* - x^*, x_2 - x_1\rangle = r\langle y^* - x^*, z\rangle,$$

which yields $\sigma_{\partial f(U)}(z) - \alpha < \langle x^*, z\rangle \leq \langle y^*, z\rangle$. Thus, the inclusion $\partial f(x_2) \subseteq Q$ holds. Using the $w^*$-openness of $Q$ and the fact that $\partial f$ is norm-$w^*$-usc at $x_2$, we get that there exists a neighborhood $V \in \mathcal{N}_X(x_2)$ with $V \subseteq U$ such that $\partial f(V) \subseteq Q$, thus $\partial f(V) \subseteq \partial f(U) \cap \{x^* \in X^* \ : \ \langle x^*, z\rangle > \sigma_{\partial f(U)}(z) - \alpha\} = S$, and so $x_2 \in G_n$.

Since $x$ and $U$ are arbitrary, we conclude that $G_n$ is dense, which allows us to conclude that $G$ is a $G_\delta$-dense subset of $W$. Using proposition 2.1.5, we get that $G \subseteq \mathfrak{D}(f)$, and so $\mathfrak{D}(f)$ is dense in $W = \operatorname{int}(\operatorname{dom} f)$. The proof is now complete. □

**Theorem 2.2.5** *Let $X$ be a Banach space. The following assertions are equivalent:*

*(a) $X$ is an Asplund space.*

*(b) For every function $f \in \Gamma_0(X)$ with $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$, the set $\mathfrak{D}(f)$ is nonempty.*

*(c) For every continuous function $f \in \Gamma_0(X)$, $\mathfrak{D}(f)$ is dense in $\operatorname{int}(\operatorname{dom} f)$.*

*(d) For every real-valued convex continuous function $f : X \to \mathbb{R}$, the set $\mathfrak{D}(f)$ is dense in $X$.*

*(e) For every $w^*$-compact set $K$ in $X^*$, the set $\mathfrak{D}(\sigma_K)$ is dense in $X$.*

*(f) For every equivalent norm $p$ on $X$, the set $\mathfrak{D}(p)$ is dense in $X$.*

*(g) For every equivalent norm $p$ on $X$, $p$ is F-differentiable at some point of $X \setminus \{0\}$.*

*Proof.* The implications $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g)$ are direct. To show that $(b) \Rightarrow (c)$ simply argue by absurd: Assume that $(b)$ holds and that there exists a proper continuous convex function $f : X \to \mathbb{R}_\infty$ such that $\mathfrak{D}(f)$ is not dense in $\operatorname{int}(\operatorname{dom} f)$. Then, there exists a convex open set $U \subseteq \operatorname{dom} f$, on which $f$ is not F-differentiable at any point $u \in U$. Finally consider $g := f + I_{\overline{U}}$, which belongs to $\Gamma_0(X)$ and also satisfies that $\mathfrak{D}(g) = \emptyset$, which contradicts $(b)$. Thus, the implication $(b) \Rightarrow (c)$ holds.

To prove $(g) \Rightarrow (a)$, assume that $(a)$ doesn't hold. Then, by Lemma 2.2.4 there exists a bounded set $K$ of $X^*$ and a real $\varepsilon > 0$, such that all $w^*$-slices of $K$ have diameter greater than $\varepsilon$. Consider now the set

$$B = \overline{\mathrm{co}}^{w^*}\left(\mathbb{B}_{X^*} + \overline{\mathrm{co}}^{w^*}(K \cup (-K))\right).$$

Noting that $\sigma_B = \|\cdot\| + \max\{\sigma_K, \sigma_{-K}\}$, it is not hard to see that for each $x \in X \setminus \{0\}$ and each $\alpha > 0$,

$$S(B,x,\alpha) \supseteq x^* + S(K,x,\alpha) \quad \text{or} \quad S(B,x,\alpha) \supseteq x^* + S(-K,x,\alpha),$$

where $x^* \in \mathbb{S}_{X^*}$ is such that $\langle x^*, x\rangle = \|x\|$. We conclude that all slices of $B$ also have diameter greater than $\varepsilon$. By construction, it is clear that $B$ is $w^*$-closed, and so its Minkowski functional $\rho_B$ is an equivalent norm on $X^*$ which is $w^*$-lower semicontinuous. Thus, $B$ is the unit ball of an equivalent dual norm and hence $p := \sigma_B$ is an equivalent norm on $X$. Then, by hypothesis, there exist a point $x \in X \setminus \{0\}$ such that $p$ is F-differentiable at $x$. According to Proposition 1.2.26, we have that $\nabla p(x) \in B$. Furthermore, it is clear that for each $\alpha > 0$, we have the inclusions

$$\nabla p(x) \in S(B,x,\alpha) \subseteq \overline{S}(B,x,\alpha) = \partial_\alpha p(x).$$

Thus, for every $n \in \mathbb{N}$ we can choose $x_n^* \in S(B,x,1/n)$ such that $\|x_n^* - \nabla p(x)\| \geq \varepsilon/2$. Using Brønsted-Rockafellar variational principle for each $n \in \mathbb{N}$, there exist $x_n \in X$ and $y_n^* \in B$ such that $y_n^* \in \partial p(x_n)$ and

$$\|x_n - x\| \leq \frac{1}{\sqrt{n}} \qquad \text{and} \qquad \|y_n^* - x_n^*\| \leq \frac{1}{\sqrt{n}}.$$

Since $\partial p$ is norm-norm-usc at $x$, there exists $n_0 \in \mathbb{N}$ such that $\|y_n^* - \nabla p(x)\| \leq \varepsilon/8$ for all $n \geq n_0$. Choosing $n$ large enough, we may suppose that $1/\sqrt{n} < \varepsilon/8$ and so,

$$\varepsilon/2 \leq \|x_n^* - \nabla p(x)\| \leq \|x_n^* - y_n^*\| + \|y_n^* - \nabla p(x)\| \leq \varepsilon/4,$$

which is a contradiction. This proves that $(g) \Rightarrow (a)$, finishing the proof. □

There is a well-known characterization of separable Asplund spaces in terms of their dual spaces (already used in Example 2.2.2(b)). The proof is not easy, but we refer the reader to [36, Theorem 2.19].

**Theorem 2.2.6** *Let $X$ be a separable Banach space. Then, $X$ is an Asplund space if and only if $X^*$ is separable.*

Next proposition establishes the stability of Asplund spaces for closed subspaces. This property is by no means trivial and partially allows us to derive the next Theorem 2.2.8. Here we follow the proof in [36, Proposition 2.33].

**Proposition 2.2.7** *Every closed subspace of an Asplund space is also an Asplund space.*

*Proof.* Let $X$ be an Asplund space, and consider a closed subspace $M$ of $X$. Reasoning by absurd, assume that $M$ is not an Asplund space, and so, in view of Lemma 2.2.4, there exists a bounded subset $K$ of $M^* = X^*/M^\perp$ and a real $\varepsilon > 0$, such that

$$\forall x \in X, \forall \alpha > 0 \ \operatorname{diam}(S(K, x, \alpha)) \geq 2\varepsilon.$$

Without loss of generality, we may assume that $K$ is $w^*$-compact and convex, just replacing $K$ by $\overline{\text{co}}^{w^*}(K)$ and noting that for all $x \in X$ and all $\alpha > 0$, $S(K, x, \alpha) \subseteq S(\overline{\text{co}}^{w^*}(K), x, \alpha)$. Let $\pi : X^* \to M^*$ the quotient map, which is known to be of norm one, an open mapping and $w^*$-$w^*$-continuous. In particular, $\pi(\mathbb{B}_{X^*})$ is a neighborhood of 0 in $M^*$.

Now, since $K$ is bounded, there exists $\lambda > 0$ such that $\pi(\lambda\mathbb{B}_{X^*})$ contains $K$. Let us denote $C := (\lambda\mathbb{B}_{X^*}) \cap \pi^{-1}(A)$. Clearly, $C$ is $w^*$-compact in $X^*$, convex and $\pi(C) = K$. Using Zorn's Lemma, there exists a minimal set $C_0$ (under inclusion) enjoying these last three properties. Since $X$ is an Asplund space, by Lemma 2.2.4 there exist $x \in X \setminus \{0\}$ and $\alpha > 0$ such that $\operatorname{diam}(S(C_0, x, \alpha)) < \varepsilon$. To simplify notation, denote $S := S(C_0, x, \alpha)$.

Since $S$ is relatively $w^*$-open in $C_0$ and it is a slice, it is not hard to see that $C_0 \setminus S$ is $w^*$-compact and convex, and so $K_0 := \pi(C_0 \setminus S)$ is $w^*$-compact and convex. Also, by minimality of $C_0$, we have that $K_0 \subsetneq K$. Now, for any two points $y_1^*, y_2^* \in K \setminus K_0$, we have that there exist $x_1^*, x_2^* \in S$ such that $\pi(x_i^*) = y_i^*$ for $i = 1, 2$. Then,

$$\|y_1^* - y_2^*\| = \|\pi(x_1^* - x_2^*)\| \leq \|x_1^* - x_2^*\| \leq \varepsilon.$$

Thus, choosing $x^* \in K_0$ and using the Hahn-Banach separation theorem for the $w^*$-topology, there exists $x' \in M$ and $\alpha' > 0$ such that

$$\langle x^*, x' \rangle > \alpha' \geq \langle x^*, y \rangle, \quad \forall y \in K \setminus K_0.$$

Thus $S(K, x', \alpha')$ separates $x^*$ and $K \setminus K_0$. Since $S(K, x', \alpha') \cap (K \setminus K_0) = \emptyset$ and $K = (K \setminus K_0) \cup K_0$, this yields the inclusion $S(K, x', \alpha') \subseteq K_0$, and so

$$\operatorname{diam}(S(K, x', \alpha')) \leq \varepsilon,$$

leading to a contradiction. The proof is then complete. □

We finish this section by presenting a final characterization of Asplund spaces that is known as separable reduction: To know whether a Banach space is an Asplund space, it is enough to check it for its separable subspaces. The reader can find the proof of the sufficiency of the following theorem in [36, Theorem 2.14], while the necessity comes directly from Proposition 2.2.7 above.

**Theorem 2.2.8** (Separable Reduction of Asplund Spaces) *Let $X$ be a Banach space. Then, $X$ is an Asplund space if and only if every separable closed subspace of $X$ is an Asplund space.*

### 2.2.2 The RNP and Stegall's Theorem

**Definition 2.2.9** (Exposed and Strongly-Exposed points) *Let $X$ be a Banach space and $K$ be a closed convex set of $X$. A point $x_0 \in K$ is said to be*

*(a)* exposed, *if there exists a functional $x^* \in X^* \setminus \{0\}$ such that*

$$\langle x^*, x_0 \rangle = \sigma_K(x^*) > \langle x^*, y \rangle, \quad \forall y \in K \setminus \{x_0\}.$$

*In such a case, we say that the functional $x^*$* exposes $x_0$ in $K$. *We will denote by* $\exp(K)$ *the set of exposed points of $K$.*

*(b)* strongly-exposed, *if there exists a functional $x^* \in X^* \setminus \{0\}$ such that $x^*$ exposes $x_0$ in $K$ and also for every sequence $(x_n) \in K$ we have that*

$$\langle x^*, x_n \rangle \to \sigma_K(x^*) \implies \|x_0 - x_n\| \to 0.$$

*In such a case, we say that $x^*$* strongly exposes $x_0$ in $K$. *We will denote by* str-exp$(K)$ *the set of strongly-exposed points of $K$.*

*Analogously, for a convex $w^*$-closed subset $K$ of $X^*$, a point $x_0^* \in K$ is said to be* $w^*$-exposed (respectively, $w^*$-strongly-exposed) *if there exists a functional $x \in X \setminus \{0\}$ which exposes (resp. strongly exposes) $x_0^*$ in $K$.*

*In such a case, we say that $x$ $w^*$-exposes (resp. $w^*$-strongly exposes) $x_0^*$ in $K$ to emphasize that the exposing functional belongs to the primal space $X$.*

**Definition 2.2.10** (Radon-Nikodým Property) *A Banach space $X$ has the Radon-Nikodým property (RNP) if every closed convex bounded subset $K$ of $X$ coincides with the closed convex hull of its strongly-exposed points.*

**Example 2.2.11** (Spaces with the RNP) Classic examples (see, e.g., [9] and [21]) are the following ones:

(a) All reflexive spaces have the RNP.

(b) $\ell^1$ has the RNP. $c_0$ and $\ell^\infty$ lack it.

(c) $L^1[0,1]$ lacks the RNP.

Next proposition establishes the first link between Asplund spaces and the RNP via *slices of small diameter*. The necessity of Proposition 2.2.12 is based on the following observation: For a closed convex bounded set $K$, it is not hard to prove from the definition of strongly-exposed points that, whenever $x \in$ str-exp$(K)$ and it is strongly-exposed by $x^* \in X^*$, then the slices $S(K, x^*, \alpha)$ can be (for some $\alpha > 0$ small enough) of diameter arbitrarily small. The sufficiency is more delicate: we refer the reader to [9, Theorem 3.5.4] or [23, Theorem 11.3] for its proof.

**Proposition 2.2.12** *A Banach space $X$ has the RNP if and only if for every nonempty bounded subset $K$ of $X$ and every $\varepsilon > 0$, there exists a slice $S(K, x^*, \alpha)$ of $K$ with diameter less than $\varepsilon$.*

We finally present one of the most celebrated results about the Radon-Nikodým property: Its duality with Asplund spaces. This theorem was first proved by C. Stegall in 1978 (see [45]), based on an earlier contribution of himself ([44]). The proof now can be found in several books, as [23, Theorem 11.8], [36, Ch. 5] or [9, Proposition 5.7.2].

**Theorem 2.2.13** (Stegall, 1978) *A Banach space $X$ is an Asplund space if and only if $X^*$ has the RNP.*

### 2.2.3 $w^*$-Asplund spaces and Collier's Theorem

A natural question concerning Stegall's theorem is whether the RNP in the primal space implies that the dual is an Asplund space. Intuitively, this shouldn't be so, since the primal space $X$ only gives enough information about conjugate functions, but doesn't describe all functions in $\Gamma_0(X)$. Form this observation is natural to introduce a "weak-star version" of Asplund spaces.

**Definition 2.2.14** ($w^*$-Asplund spaces) *Let $X$ be a Banach space. We say that the dual space $X^*$ is a* $w^*$-Asplund space *if for every function $f \in \Gamma_0(X^*, w^*)$, the set $\mathfrak{D}(f)$ of points where $f$ is Fréchet-differentiable is dense in* $\operatorname{int}(\operatorname{dom} f)$.

In 1976, J. Collier (see [14]) characterized the Banach spaces with the RNP as those whose dual is a $w^*$-Asplund space. This contribution completed the framework of duality between Asplund spaces and RNP spaces: When the primal space has the RNP we can only assure the Fréchet-differentiability of conjugate functions. We refer the reader to [14] for the original proof and to [9, Theorem 5.7.4] for an alternative one.

**Theorem 2.2.15** (Collier, 1976) *A Banach space $X$ has the RNP if and only if its dual space $X^*$ is a $w^*$-Asplund space.*

The next known Theorem 2.2.18 gives various characterizations of $w^*$-Asplund spaces. Because of their importance in the development of this chapter (besides Theorem 2.2.5), we provide a proof of the theorem. The proof below will involve the aforementioned Collier's theorem. In a first step the two following known lemmas are needed. Both lemmas are hard to find in their present form, but can be retrieved by using different proposition of the books [9], [21] or [36]. We establish their proof for completeness and convenience of the reader.

**Lemma 2.2.16** *Let $K$ be a nonempty closed convex bounded set of $X$, $x_0 \in K$ and*

$x_0^* \in X^* \setminus \{0\}$. *The following assertions are equivalent:*

*(a)* $x_0^*$ *strongly exposes* $x_0$ *in* $K$.

*(b)* $x_0^*$ $w^*$*-strongly exposes* $x_0$ *in* $\overline{K}^{w^{**}}$ *(as a subset of* $X^{**}$*).*

*(c)* $\sigma_K$ *is F-differentiable at* $x_0^*$ *with* $\nabla\sigma_K(x_0^*) = x_0$.

*Proof.* Let us prove first $(c) \Rightarrow (a)$. Assume then that $\sigma_K$ is F-differentiable at $x_0^*$ with $\nabla\sigma_K(x_0^*) = x_0$, and choose a sequence $(x_n) \subseteq K$ such that $\langle x_0^*, x_n\rangle \to \sigma_K(x_0^*)$. It is not hard to realize that there exists a sequence $(\varepsilon_n) \subseteq ]0, +\infty[$ converging to 0 such that $x_n \in \partial_{\varepsilon_n}\sigma_K(x_0^*)$, and so, $x_0^* \in \partial_{\varepsilon_n} I_K(x_n)$. For each $n \in \mathbb{N}$, applying the Brønsted-Rockafellar variational principle (see Theorem 2.1.6) with $\lambda = \sqrt{\varepsilon_n}$, we find a sequence $(y_n^*, y_n) \in \partial\sigma_K$ such that

$$\|y_n - x_n\| \leq \sqrt{\varepsilon_n} \qquad \text{and} \qquad \|y_n^* - x_0^*\| \leq \sqrt{\varepsilon_n}.$$

In particular, $y_n^* \to x_0^*$. Using Proposition 2.1.5, we get that $y_n \to x_0$ and so

$$\|x_n - x_0\| \leq \|y_n - x_n\| + \|y_n - x_0\| \to 0.$$

Since $(x_n)$ is an arbitrary sequence, it follows that $x_0^*$ strongly exposes $x_0$ in $K$.

Let us prove now $(a) \Rightarrow (b)$. Fix $\varepsilon > 0$. Since $x_0^*$ strongly exposes $x_0$, it is not hard to deduce that there exists $\alpha > 0$ such that

$$S(K, x_0^*, \alpha) \subseteq \partial_\alpha\sigma_K(x_0^*) \subseteq B_X(x_0, \varepsilon) \cap K.$$

Now, let $x^{**} \in S\left(\overline{K}^{w^{**}}, x_0^*, \alpha\right)$. Then, there exists a net $(x_i)_{i\in I}$ included in $\partial_\alpha\sigma_K(x_0^*)$ such that $x_i \rightharpoonup^{w^{**}} x^{**}$. In particular, for each $x^* \in \mathbb{S}_{X^*}$ we have that

$$\langle x^*, x^{**} - x_0\rangle = \lim_{i\in I}\langle x^*, x_i - x_0\rangle \leq \varepsilon.$$

Thus, $x^{**} \in B_{X^{**}}(x_0, \varepsilon) \cap \overline{K}^{w^{**}}$, and so we get

$$S\left(\overline{K}^{w^{**}}, x_0^*, \alpha\right) \subseteq B_{X^{**}}(x_0, \varepsilon) \cap \overline{K}^{w^{**}}.$$

Since $\varepsilon$ is arbitrary, we conclude that the slices induced by $x_0^*$ form a neighborhood basis of $x_0$ for the strong topology relative to $\overline{K}^{w^{**}}$. This entails that $x_0$ is $w^*$-strongly exposed by $x_0^*$ in $\overline{K}^{w^{**}}$, proving the desired implication.

It only rests to show that $(b) \Rightarrow (c)$. Since $x_0$ is $w^*$-strongly exposed in $\overline{K}^{w^{**}}$ by $x_0^*$, we have in particular that $\partial\sigma_K(x_0^*) = \{x_0\}$. Now, since $\overline{K}^{w^{**}}$ is bounded, we have that $\sigma_K$ is continuous and so, according to Proposition 2.1.5, to prove that $\sigma_K$ is F-differentiable at $x_0^*$ is enough to show that $\partial\sigma_K$ is norm-norm-usc at $x_0^*$. Fix then a sequence $(x_n^*)$ in

$X^*$ converging to $x_0^*$ and for each $n \in \mathbb{N}$, choose $x_n^{**} \in \partial\sigma_K(x_n^*)$. We then have that $x_n^{**} \in \overline{K}^{w^{**}}$ and that $\langle x_n^{**}, x_n^* \rangle = \sigma_K(x_n^*)$. Thus, we can write

$$\begin{aligned} |\langle x_n^{**} - x_0, x_0^* \rangle| &= |\langle x_n^{**} - x_0, x_0^* - x_n^* \rangle + \langle x_n^{**}, x_n^* \rangle - \langle x_0, x_n^* \rangle| \\ &\leq \|x_n^{**} - x_0\| \|x_0^* - x_n^*\| + \sigma_K(x_n^*) - \langle x_0, x_n^* \rangle| \\ &\longrightarrow 0, \end{aligned}$$

since the sequence $(\|x_n^{**} - x_0\|)$ is bounded, along with $\sigma_K(x_n^*) \to \sigma_K(x_0^*) = \langle x_0^*, x_0 \rangle$ and $\langle x_0, x_n^* \rangle \to \langle x_0, x_0^* \rangle$. This yields that $\langle x_n^{**}, x_0^* \rangle \to \langle x_0, x_0^* \rangle = \sigma_K(x_0^*)$, and since $x_0^*$ $w^*$-strongly exposes $x_0$ in $\overline{K}^{w^{**}}$, we deduce that $x_n^{**} \to x_0$. We then conclude that $\partial\sigma_K$ is norm-norm-usc at $x_0^*$, finishing the proof. □

**Lemma 2.2.17** *A Banach space $X$ has the RNP if and only if for every equivalent norm $p$ on $X$,* str-exp $\left(\mathbb{B}_{(X,p)}\right) \neq \emptyset$.

*Proof.* Since the necessity is direct, we only need to prove the sufficiency. Reasoning by absurd, assume that $X$ lacks the RNP. Then, according to Proposition 2.2.12, there exists a nonempty bounded set $K$ and a real $\varepsilon > 0$ such that $K$ has all slices with diameter grater than $\varepsilon$. Following the same construction than in the proof of Theorem 2.2.5, we get that the set

$$B = \overline{\text{co}} \left( \mathbb{B}_{X^*} + \overline{\text{co}}(K \cup (-K)) \right)$$

has all its slices with diameter greater than $\varepsilon$. This entails that str-exp$(B) = \emptyset$, which is a contradiction since the Minkowski functional $\rho_B$ is an equivalent norm on $X$ and $\mathbb{B}_{(X,\rho)} = B$. □

**Theorem 2.2.18** *Let $X$ be a Banach space. The following assertions are equivalent:*

*(a) $X^*$ is a $w^*$-Asplund space.*

*(b) For every function $f \in \Gamma_0(X^*, w^*)$ with* $\text{int}(\text{dom}\, f) \neq \emptyset$, *the set $\mathfrak{D}(f)$ is nonempty.*

*(c) For every continuous function $f \in \Gamma_0(X^*, w^*)$, $\mathfrak{D}(f)$ is dense in* $\text{int}(\text{dom}\, f)$.

*(d) For every convex real-valued $w^*$-lower semicontinuous function $f : X^* \to \mathbb{R}$, the set $\mathfrak{D}(f)$ is dense in $X^*$.*

*(e) For every nonempty bounded subset $K$ of $X$, the set $\mathfrak{D}(\sigma_K)$ is dense in $X^*$.*

*(f) For every equivalent norm $p$ on $X$, the set $\mathfrak{D}(p_*)$ is dense in $X^*$, where $p_*$ is the dual norm on $X^*$ induced by $p$.*

*(g) For every equivalent norm $p$ on $X$, the dual norm $p_*$ is F-differentiable at some point of $X^* \setminus \{0\}$.*

*Proof.* As in Theorem 2.2.5, we only need to prove $(g) \Rightarrow (a)$. Using Lemma 2.2.16, the hypothesis implies that for every equivalent norm $p$ on $X$, its unit ball $\mathbb{B}_{(X,p)}$ has a strongly exposed point. Thus, according to Lemma 2.2.17, this yields that $X$ has the RNP. The conclusion follows by Collier's theorem. $\square$

## 2.3 General smooth-like properties for convex functions

In this section and the next one which are our contribution in this chapter, we will study the "smooth-like" properties in the convex analysis framework. Usually, when we find smooth properties of convex functions defined over a Banach space $X$, it is possible to find equivalent interpretations of them as geometrical properties of convex sets of its dual space (or predual space). To do such interpretation, we present a general strategy to reduct this kind of properties.

The first step is to formalize the idea of what a property of Banach spaces is: A property $(\mathcal{P})$ of Banach spaces will be understood as *a family of binary functions* $\{\mathcal{P}_X : D(X) \to \{0,1\}\}$ indexed by the class of Banach spaces for which the domain $D(X)$ of each function depends on the index space $X$.

The intuition of the family of functions $\{\mathcal{P}_X\}$ is that for $z \in D(X)$, $\mathcal{P}_X(z) = 1$ means that *the property* $(\mathcal{P})$ *holds at* $z$. So, in order to define a property $(\mathcal{P})$ we need to specify for each Banach space $X$ 1) the domain $D(X)$ of $\mathcal{P}_X$; and 2) what $\mathcal{P}_X(z) = 1$ means, usually by the equivalence

$$\mathcal{P}_X(z) = 1 \iff (\mathcal{P}) \text{ holds at } z.$$

Here, we will be concerned with properties of the convex smooth-like type as defined in the next subsections. This type of properties are motivated by Asplund spaces and their geometrical interpretation with the RNP (see Theorem 2.2.13).

### 2.3.1 Convex smooth-like properties

**Definition 2.3.1** *A property $(\mathcal{P})$ of Banach spaces is a* convex smooth-like property *if for each Banach space $X$, $D(X) = \Gamma_0(X) \times X$ and the function $\mathcal{P}_X$ satisfy the following conditions:*

*(i) $\mathcal{P}_X$ is **local**: For each pair of two functions $f, g \in \Gamma_0(X)$ and for each open set $U \subseteq X$ we have that*

$$f\big|_U = g\big|_U \Rightarrow \mathcal{P}_X(f,\cdot)\big|_U = \mathcal{P}_X(g,\cdot)\big|_U.$$

*(ii)* $\mathcal{P}_X$ *is* ***transitive****: For each other Banach space $Y$ such that there exists an onto bounded linear operator $T : X \to Y$, we have that for all $f \in \Gamma_0(Y)$*

$$\mathcal{P}_X(f \circ T, x) = \mathcal{P}_Y(f, Tx), \ \forall x \in X.$$

*(iii)* $\mathcal{P}_X$ *is* ***set-consistent****: For every $w^*$-closed convex set $K$ of $X^*$ we have that*

*(iii.a)* $\forall x \in X, \forall t > 0$, $\mathcal{P}_X(\sigma_K, x) = \mathcal{P}(\sigma_K, tx)$.

*(iii.b)* $\forall x^* \in X^*$, $\mathcal{P}_X(\sigma_{K+x^*}, \cdot) = \mathcal{P}_X(\sigma_K, \cdot)$.

*(iv)* $\mathcal{P}_X$ *is* ***epigraphical****: For each function $f \in \Gamma_0(X)$ and each $x \in X$,*

$$\mathcal{P}_X(f, x) = \mathcal{P}_{X\times\mathbb{R}}(\sigma_{\operatorname{epi} f^*}, (x, -1)).$$

In the case of convex smooth-like properties, $\mathcal{P}_X(f, x) = 1$ means that $f$ *satisfy the property* $(\mathcal{P})$ *at* $x$. For reducing notation, whenever there is no confusion we will omit the Banach space index, writing simply $\mathcal{P}(\cdot, \cdot)$. Also, in the case of a support function $\sigma_K$ (where $K$ is a $w^*$-closed convex set of the dual space), we will sometimes write $\mathcal{P}(K, \cdot)$ instead of $\mathcal{P}(\sigma_K, \cdot)$.

Finally, for a function $f \in \Gamma_0(X)$, we will denote by $\mathcal{P}[f] := \{x \in \operatorname{dom} f \ : \ \mathcal{P}(f, x) = 1\}$. Consistently, if $f = \sigma_K$, then we will sometimes write $\mathcal{P}[K]$ instead of $\mathcal{P}[\sigma_K]$.

**Definition 2.3.2** (($\mathcal{P}$)-structural spaces) *Let $(\mathcal{P})$ be a convex smooth-like property. A Banach space $X$ is said to be*

1. $(\mathcal{P})$***-structural*** *if for each $f \in \Gamma_0(X)$, $\mathcal{P}[f]$ is dense in* $\operatorname{int}(\operatorname{dom} f)$.

2. $(\mathcal{P})$*-$w^*$**-geometrical*** *if for each $w^*$-compact convex set $K \subseteq X^*$, $\mathcal{P}[K]$ is dense in $X$.*

**Lemma 2.3.3** ([36, Lemma 2.31]) *Let $X$ be a Banach space, $f \in \Gamma_0(X)$ and $x_0 \in$* $\operatorname{int}(\operatorname{dom} f)$. *Then, there exist a neighborhood $U \in \mathcal{N}_X(x_0)$ and a convex Lipschitz-continuous function $\tilde{f} : X \to \mathbb{R}$ such that*

$$\tilde{f}\big|_U = f\big|_U.$$

*Proof.* According to Proposition 1.2.21, there exist $M > 0$ and $U \in \mathcal{N}_X(x_0)$ included in $\operatorname{int}(\operatorname{dom} f)$ such that $\partial f(U) \subseteq M\mathbb{B}_{X^*}$. Define

$$\tilde{f} := f \square (M\|\cdot\|).$$

It is clear that $\tilde{f} \leq f$, and that it is convex. By continuity on $U$, we know that for every $u \in U$ there exists $u^* \in \partial f(u)$. Since $\|u^*\| \leq M$ by construction, we can write

$$f(u) \leq f(x) + \langle u^*, u - x\rangle \leq f(x) + M\|u - x\|, \quad \forall x \in X,$$

which entails that $\tilde{f}(u) = f(u)$ for all $u \in U$. It only rests to prove that $\tilde{f}$ is Lipschitz-continuous. Fix then $u, v \in X$. By definition of the inf-convolution, for every $\varepsilon > 0$, there exists $y \in \operatorname{dom} f$ such that

$$\tilde{f}(u) > f(y) + M\|u - y\| - \varepsilon.$$

Since $\tilde{f}(v) \leq f(y) + M\|v - y\|$, this yields

$$\tilde{f}(v) - \tilde{f}(u) \leq M\|v - y\| - M\|u - y\| + \varepsilon \leq M\|v - u\| + \varepsilon$$

By arbitrariness of $\varepsilon$, $u$ and $v$, we deduce that $\tilde{f}$ is $M$-Lipschitz on $X$. The proof is now complete. □

**Proposition 2.3.4** *Let $X$ be a Banach space. The following assertions are equivalent:*

*(a) For each $f \in \Gamma_0(X)$ with $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$, $\operatorname{int}(\operatorname{dom} f) \cap \mathcal{P}[f] \neq \emptyset$.*

*(b) $X$ is $(\mathcal{P})$-structural.*

*(c) For each continuous proper convex function $f : X \to \mathbb{R}_\infty$, $\mathcal{P}[f]$ is dense in $\operatorname{int}(\operatorname{dom} f)$.*

*(d) For each real-valued function $f \in \Gamma_0(X)$, $\mathcal{P}[f]$ is dense in $X$.*

*Proof.* The implications $(b) \Rightarrow (c) \Rightarrow (d)$ are obvious. For $(a) \Rightarrow (b)$ assume that $(a)$ holds but $X$ is not $(\mathcal{P})$-structural. Then, there exists a function $f \in \Gamma_0(X)$ and a nonempty open set $U \subseteq \operatorname{int}(\operatorname{dom} f)$, such that

$$\mathcal{P}(f, u) = 0, \ \forall u \in U.$$

Take $x \in U$ and $\varepsilon > 0$ small enough such that $B = B_X[x, \varepsilon] \subseteq U$ and consider $\tilde{f} = f + I_B$ which is a function in $\Gamma_0(X)$ with $\operatorname{int}(\operatorname{dom} \tilde{f}) = \operatorname{int}(B)$. Now, since

$$\tilde{f}\big|_{\operatorname{int}(B)} = f\big|_{\operatorname{int}(B)}$$

and $\operatorname{int}(B) \subseteq U$, we get, since $(\mathcal{P})$ is local, that $\mathcal{P}(\tilde{f}, z) = \mathcal{P}(f, z) = 0$ for each $z \in \operatorname{int}(B)$. Thus, $\operatorname{int}(\operatorname{dom} \tilde{f}) \cap \mathcal{P}[\tilde{f}] = \emptyset$ which is a contradiction.

Now, for $(d) \Rightarrow (a)$, let $f \in \Gamma_0(X)$ with $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$ and $x \in \operatorname{int}(\operatorname{dom} f)$. Applying Lemma 2.3.3, we know that there exists a convex Lipschitz-continuous function $\tilde{f} : X \to \mathbb{R}$ and an open neighborhood $U$ of $x$ such that $\tilde{f}\big|_U = f\big|_U$. In particular, $f$ is real-valued in $U$ and so $U \subseteq \operatorname{int}(\operatorname{dom} f)$. Since $\tilde{f}$ is real-valued, $\mathcal{P}[\tilde{f}]$ is dense in $X$ and then, there exists $z \in U$ such that $\mathcal{P}(\tilde{f}, z) = 1$. Finally, since $(\mathcal{P})$ is local, $\mathcal{P}(f, z) = 1$, finishing the proof. □

To get similar characterizations for $(\mathcal{P})$-$w^*$-geometrical Banach spaces, we will need to establish some lemmas.

**Lemma 2.3.5** ([9, Lemma 2.3.6]) *Let $K$ be a nonempty bounded set of $X$. For every $x^* \in X^* \setminus \{0\}$ and every $\alpha > 0$, there exists $\varepsilon > 0$ small enough such that*

$$S\left(K, y^*, \tfrac{\alpha}{2}\right) \subseteq S(K, x^*, \alpha), \quad \forall y^* \in B_{X^*}[x^*, \varepsilon].$$

*Proof.* Define $M := \sup\{\|x\| : x \in K\}$ and choose $\varepsilon \in \left]0, \frac{\alpha}{4M}\right[$, with the convention $\frac{\alpha}{4M} = +\infty$ if $M = 0$. Fix any functional $y^* \in B_{X^*}[x^*, \varepsilon]$. Now, for $z \in S\left(K, y^*, \frac{\alpha}{2}\right)$ we can write

$$\begin{aligned}
\langle x^*, z\rangle &\geq \langle y^*, z\rangle - |\langle x^* - y^*, z\rangle| \geq \sigma_K(y^*) - \frac{\alpha}{2} - \varepsilon M \\
&\geq \sup_{x\in K}\{\langle x^*, x\rangle - |\langle x^* - y^*, x\rangle|\} - \frac{\alpha}{2} - \varepsilon M \\
&\geq \sigma_K(x^*) - \frac{\alpha}{2} - 2\varepsilon M \\
&> \sigma_K(x^*) - \alpha.
\end{aligned}$$

Thus, $z \in S(K, x^*, \alpha)$, which finishes the proof. □

**Lemma 2.3.6** *Let $K$ be a $w^*$-closed convex set of $X^*$ with $\sigma_K : X \to \mathbb{R}_\infty$ its support function. Assume that* $\operatorname{int}(\operatorname{dom}\sigma_K) \neq \emptyset$ *and let* $x \in \operatorname{int}(\operatorname{dom}\sigma_K)$ *with $x \neq 0$. Then, there exists an equivalent norm $p$ on $X$, a point $x_0^* \in X^*$ and an open neighborhood $U$ of $x$ such that*

$$\sigma_{K+x_0^*}(u) = p(u), \ \forall u \in U.$$

*Proof.* Let $\eta = \sigma_K(x)$. Let us assume that $\eta > 0$ and define $K_1 = \partial_{\eta/2}\sigma_K(x) \subseteq X^*$. By Lemma 2.3.5, there exists an open convex neighborhood $U_1$ of $x$ such that $S\left(K, u, \frac{\eta}{4}\right) \subseteq K_1$ for all $u \in U_1$. This entails that for every $u \in U_1$,

$$\sigma_K(u) = \sup\left\{\langle z^*, u\rangle : z^* \in S\left(K, u, \tfrac{\eta}{4}\right)\right\} \leq \sigma_{K_1}(u),$$

and so, since $\sigma_{K_1} \leq \sigma_K$, we get that $\sigma_K\big|_{U_1} = \sigma_{K_1}\big|_{U_1}$. Also, the latter relation provides that $\partial\sigma_K(u) = \partial\sigma_{K_1}(u)$, for all $u \in U_1$.

Define now $K_2 = \overline{\operatorname{co}}^{w^*}[K_1 \cup (-K_1)]$. It is clear that $\sigma_{K_2} = \max\{\sigma_{K_1}, \sigma_{-K_1}\}$ and therefore, since $\langle z^*, x\rangle \geq \eta - \frac{\eta}{2} = \frac{\eta}{2}$ for all $z^* \in K_1$, we have that

$$\sigma_{-K_1}(x) = \sup_{z^*\in K_1}\langle -z^*, x\rangle \leq -\frac{\eta}{2} < \frac{\eta}{2} \leq \sigma_{K_1}(x).$$

Thus, $\sigma_{-K_1}(x) < \sigma_{K_1}(x)$ and, by continuity, there exists a neighborhood $U_2$ of $x$ included in $U_1$ such that $\sigma_{-K_1}\big|_{U_2} < \sigma_{K_1}\big|_{U_2}$. In particular, we get that $\sigma_{K_2}\big|_{U_2} = \sigma_{K_1}\big|_{U_2}$. Finally, define the set $B = \frac{\eta}{2\|x\|}\mathbb{B}_{X^*}$ and consider the set $K_3 = \overline{\operatorname{co}}^{w^*}[K_2 \cup B]$. Again, we have that

$$\sigma_B(x) = \frac{\eta}{2\|x\|}\|x\| = \frac{\eta}{2} < \eta = \sigma_{K_2}(x),$$

and so, by continuity, there exists a neighborhood $U_3$ of $x$ included in $U_2$ such that $\sigma_{K_3}\big|_{U_3} = \sigma_{K_2}\big|_{U_3}$. Let $U = U_3$ and $p = \sigma_{K_3}$. It is clear by the construction of $K_3$ that $p$ is an equivalent norm on $X$ and $K_3 = \mathbb{B}_{(X^*,p_*)}$. Moreover, $\sigma_K\big|_U = p\big|_U$ and so, the conclusion holds with $x_0^* = 0$.

If $\eta \leq 0$, we can choose $x_0^* \in X^*$ such that $\langle x_0^*, x\rangle > -\eta$, and repeat the previous procedure with $K + x_0^*$ instead of $K$, which finishes the proof. □

**Theorem 2.3.7** *The following assertions are equivalent:*

*(a) For each $w^*$-closed convex set $K \subseteq X^*$ with $\operatorname{int}(\operatorname{dom}\sigma_K) \neq \emptyset$,*

$$\mathcal{P}[\sigma_K] \cap \operatorname{int}(\operatorname{dom}\sigma_K) \neq \emptyset.$$

*(b) For each $w^*$-closed convex set $K \subseteq X^*$, $\mathcal{P}[\sigma_K]$ is dense in $\operatorname{int}(\operatorname{dom}\sigma_K)$.*

*(c) $X$ is $(\mathcal{P})$-$w^*$-geometrical.*

*(d) For each equivalent norm $p$ on $X$, $\mathcal{P}[p]$ is dense in $X$.*

*Proof.* $(b)\Rightarrow(c)\Rightarrow(d)$ are obvious. For $(d)\Rightarrow(a)$, assume that $(d)$ holds and that there exists a $w^*$-closed convex set $K \subseteq X^*$ with $\operatorname{int}(\operatorname{dom}\sigma_K) \neq \emptyset$ such that $\mathcal{P}[\sigma_K]\cap\operatorname{int}(\operatorname{dom}\sigma_K) = \emptyset$. Choose $x \in \operatorname{int}(\operatorname{dom}\sigma_K)$. Applying Lemma 2.3.6, we get that there exists $x_0^* \in X^*$, an equivalent norm $p$ on $X$ and an open neighborhood $U$ of $x$, which we can assume contained in $\operatorname{int}(\operatorname{dom}\sigma_K)$, such that

$$\sigma_{K+x_0^*}\big|_U = p\big|_U.$$

Since $(\mathcal{P})$ is local, $\mathcal{P}(\sigma_{K+x_0^*}, u) = \mathcal{P}(p, u)$ for all $u \in U$ and, since $(\mathcal{P})$ is set-consistent, we conclude that

$$\mathcal{P}(p, u) = \mathcal{P}(\sigma_K, u) = 0, \ \forall u \in U,$$

which is clearly a contradiction, according to the density of $\mathcal{P}[p]$ in $X$.

To prove $(a)\Rightarrow(b)$, assume that $(a)$ holds and there exists a $w^*$-closed convex set $K \subseteq X^*$ and an open set $U \subseteq \operatorname{int}(\operatorname{dom}\sigma_K)$ such that $\mathcal{P}[\sigma_K] \cap U = \emptyset$. Choose $x \in U$ with $x \neq 0$ and $\delta \in \left]0, \frac{1}{2}\|x\|\right[$ small enough such that $x + \delta\mathbb{B}_X \subseteq U \setminus \{0\}$ and define

$$C = \operatorname{cone}(x + \delta\mathbb{B}_X) := \mathbb{R}_+ (x + \delta\mathbb{B}_X).$$

Since $\|u\| \geq \frac{1}{2}\|x\| > 0$, it is not hard to see that $C$ is a closed convex set and that $\mathcal{P}(\sigma_K, z) = 0$ for all $z \in \operatorname{int}(C)$ (since $(\mathcal{P})$ is set-consistent). Let us consider the function $\sigma = \sigma_K + I_C$. Since $I_C$ is a sublinear lsc function, we get $\sigma$ is also a sublinear lsc function. By Proposition 1.2.26, $\sigma = \sigma_{K'}$, where $K' := \partial\sigma(0)$ is a $w^*$-closed convex set of $X^*$. Thus, we can write

$$\sigma_K(z) = \sigma_{K'}(z), \quad \forall z \in \operatorname{int}(C)$$

and so, since $(\mathcal{P})$ is local, we get that $\mathcal{P}(\sigma_{K'}, z) = 0$ for all $z \in \operatorname{int}(C)$. Noting that $\operatorname{dom} \sigma_{K'} = C$, we conclude that $\mathcal{P}[\sigma_{K'}] \cap \operatorname{int}(\operatorname{dom} \sigma_{K'}) = \emptyset$, which is a contradiction. The proof is now complete. $\square$

Now, we would like to link Proposition 2.3.4 and Theorem 2.3.7. To do so, we will use the fact that $(\mathcal{P})$ is epigraphical and the following simple lemma:

**Lemma 2.3.8** *Let $\theta$ be a (Hausdorff) locally convex topology on $X$ coarser than $\tau_{\|\cdot\|}$. For a function $f \in \Gamma_0(X, \theta)$ we have that*

*(a)* $\operatorname{int}(\operatorname{dom} \sigma_{\operatorname{epi} f^*}) = ]0, +\infty[ \cdot (\operatorname{int}(\operatorname{dom} f) \times \{-1\})$.

*(b)* $\operatorname{int}(\operatorname{dom} f) = \{x \in X \ : \ (x, -1) \in \operatorname{int}(\operatorname{dom} \sigma_{\operatorname{epi} f^*})\}$.

*where the interior is considered for the topology $\tau_{\|\cdot\|}$ and the conjugate $f^*$ is taken with respect to the duality $\langle X, (X, \theta)^* \rangle$.*

*Proof.* Note first that $(b)$ is directly deduced from $(a)$. Now, to prove $(a)$, let us show both inclusions:

- Fix $(x, s) \in \operatorname{int}(\operatorname{dom} \sigma_{\operatorname{epi} f^*})$. Since $\sigma_{\operatorname{epi} f^*}(x', s') = +\infty$ whenever $s' > 0$, it is easy to deduce that $s < 0$. Thus, denoting $\bar{x} := |s|^{-1} x$, we have $(\bar{x}, -1) \in \operatorname{int}(\operatorname{dom} \sigma_{\operatorname{epi} f^*})$. Also, there exists a neighborhood $U \in \mathcal{N}_X(\bar{x})$ such that for all $u \in U$, $(u, -1) \in \operatorname{int}(\operatorname{dom} \sigma_{\operatorname{epi} f^*})$. We then conclude that, for each $u \in U$

$$+\infty > \sigma_{\operatorname{epi} f^*}(u, -1) = \sup_{(y^*, r) \in \operatorname{epi} f^*} \{\langle y^*, u \rangle - r\} = \sup_{y^* \in \operatorname{dom} f^*} \{\langle y^*, u \rangle - f^*(y^*)\} = f(u).$$

  This yields $U \subseteq \operatorname{dom} f$, and so $(\bar{x}, -1) \in \operatorname{int}(\operatorname{dom} f) \times \{-1\}$, proving that $(x, s) \in ]0, +\infty[ \cdot (\operatorname{int}(\operatorname{dom} f) \times \{-1\})$. We conclude that

$$\operatorname{int}(\operatorname{dom} \sigma_{\operatorname{epi} f^*}) \subseteq ]0, +\infty[ \cdot (\operatorname{int}(\operatorname{dom} f) \times \{-1\}),$$

  proving our first inclusion.

- To prove the reverse inclusion, it is enough to show that $\operatorname{int}(\operatorname{dom} f) \times \{-1\} \subseteq \operatorname{dom} \sigma_{\operatorname{epi} f^*}$, since $\operatorname{dom} \sigma_{\operatorname{epi} f^*}$ is a cone and that $]0, +\infty[ \cdot (\operatorname{int}(\operatorname{dom} f) \times \{-1\})$ is an open set in $X \times \mathbb{R}$. Fix then $x \in \operatorname{int}(\operatorname{dom} f)$. Fixing $\varepsilon > 0$, we know by Proposition 1.2.24 that there exists $x^* \in \partial_\varepsilon f(x)$, where the subdifferential is taken with respect to the duality $\langle X, (X, \theta)^* \rangle$. Thus, according again to Proposition 1.2.24, we deduce that

$$\sup_{(y^*, r) \in \operatorname{epi} f} \{\langle y^*, x \rangle - r\} = f^{**}(x) = f(x) \leq \langle x^*, x \rangle - f^*(x^*) + \varepsilon.$$

  This yields to $\sigma_{\operatorname{epi} f^*}(x, -1) < +\infty$, showing that $(x, -1) \in \operatorname{dom} \sigma_{\operatorname{epi} f^*}$ and finishing the proof.

$\square$

**Proposition 2.3.9** *A Banach space $X$ is $(\mathcal{P})$-structural if and only if $X \times \mathbb{R}$ is $(\mathcal{P})$-$w^*$-geometrical.*

*Proof.* Le us show first the sufficiency. Assume that $X \times \mathbb{R}$ is $(\mathcal{P})$-$w^*$-geometrical and let $f \in \Gamma_0(X)$ with $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$. By Lemma 2.3.8(a), we know that $\operatorname{int}(\operatorname{dom} \sigma_{\operatorname{epi} f^*})$ is nonempty. As $X \times \mathbb{R}$ is $(\mathcal{P})$-$w^*$-structural, equivalence $(a) \Leftrightarrow (c)$ in Theorem 2.3.7 ensures that $\mathcal{P}[\operatorname{epi} f^*] \cap \operatorname{int}(\operatorname{dom} \sigma_{\operatorname{epi} f^*}) \neq \emptyset$. Since the second component of all points in $\operatorname{int}(\operatorname{dom} \sigma_{\operatorname{epi} f^*})$ are negative by Lemma 2.3.8(a) again and since $(\mathcal{P})$ is set-consistent, there exists $x \in X$ such that

$$(x, -1) \in \mathcal{P}[\operatorname{epi} f^*] \cap \operatorname{int}(\operatorname{dom} \sigma_{\operatorname{epi} f^*}).$$

By Lemma 2.3.8.(b), we get that $x \in \operatorname{int}(\operatorname{dom} f)$ and, since $(\mathcal{P})$ is epigraphical, $\mathcal{P}(f, x) = 1$. The conclusion follows from the equivalence $(a) \Leftrightarrow (b)$ in Proposition 2.3.4.

Assume now that $X$ is $(\mathcal{P})$-structural and let $p$ be an equivalent norm over $X \times \mathbb{R}$. Consider the function $\varphi : X^* \to \mathbb{R}_\infty$ given by

$$\varphi(x^*) = \inf\{t \;:\; p_*(x^*, t) \leq 1\}.$$

It is easy to see that $\operatorname{epi} \varphi = \mathbb{B}_{(X^* \times \mathbb{R}, p_*)} + L$ where $L = \{(0, t) \;:\; t \geq 0\}$. Since $\mathbb{B}_{(X^* \times \mathbb{R}, p_*)}$ is $w^*$-compact and convex and $L$ is $w^*$-closed and convex, we get that $\operatorname{epi} \varphi$ is $w^*$-closed and convex. Thus, there exists $f \in \Gamma_0(X)$ such that $f^* = \varphi$. Note also that

$$\forall (x, \lambda) \in X \times ]-\infty, 0[, \;\; \sigma_{\operatorname{epi} \varphi}(x, \lambda) = \sigma_{\mathbb{B}_{(X^* \times \mathbb{R}, p_*)}}(x, \lambda) + \sigma_L(x, \lambda) = \sigma_{\mathbb{B}_{(X^* \times \mathbb{R}, p_*)}}(x, \lambda), \quad (2.8)$$

so $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$ by Lemma 2.3.8, and hence $\mathcal{P}[\operatorname{epi} f^*]$ is dense in $\operatorname{int}(\operatorname{dom} f)$ by the $(\mathcal{P})$-structural property of $X$. Using that $(\mathcal{P})$ is epigraphical and set-consistent and applying Lemma 2.3.8, we can conclude that $\mathcal{P}[\operatorname{epi} \varphi]$ is dense in $\operatorname{int}(\operatorname{dom} \sigma_{\operatorname{epi} \varphi}) = X \times ]-\infty, 0[$.

Further, according to equation (2.8) and since $(\mathcal{P})$ is local, we get that $\mathcal{P}[\mathbb{B}_{(X^* \times \mathbb{R}, p_*)}]$ is dense in $X \times ]-\infty, 0[$. To conclude, consider the automorphism $T : X \times \mathbb{R} \to X \times \mathbb{R}$ given by $T(x, t) = (x, -t)$, and repeat the same argument for the equivalent norm $p \circ T$ over $X \times \mathbb{R}$. We then get that $\mathcal{P}[p \circ T]$ is dense in $X \times ]-\infty, 0[$. Now, since $(\mathcal{P})$ is transitive,

$$\forall (x, t) \in X \times \mathbb{R}, \;\; \mathcal{P}(p \circ T, (x, t)) = \mathcal{P}(p, T(x, t)).$$

In particular, since $p$ is the support function of $\mathbb{B}_{(X^* \times \mathbb{R}, p_*)}$, we have that $\mathcal{P}[\mathbb{B}_{(X^* \times \mathbb{R}, p_*)}] = \mathcal{P}[p]$ and therefore $\mathcal{P}[\mathbb{B}_{(X^* \times \mathbb{R}, p_*)}]$ is dense in $T\big(X \times ]-\infty, 0[\big) = X \times ]0, \infty[$. We deduce that $\mathcal{P}[\mathbb{B}_{(X^* \times \mathbb{R}, p_*)}]$ is dense in $X \times \mathbb{R}$, proving that $X \times \mathbb{R}$ is $(\mathcal{P})$-$w^*$-geometrical (according to equivalence $(b) \Leftrightarrow (d)$ in Theorem 2.3.7). $\square$

The following proposition establishes the natural stability results for $(\mathcal{P})$-structural and $(\mathcal{P})$-$w^*$-geometrical spaces, due to transitivity.

**Proposition 2.3.10** *Let $X, Y$ be two Banach spaces with an onto bounded linear operator $T : X \to Y$. Then,*

1. *$X$ is $(\mathcal{P})$-$w^*$-geometrical $\Rightarrow$ $Y$ is $(\mathcal{P})$-$w^*$-geometrical.*

2. *$X$ is $(\mathcal{P})$-structural $\Rightarrow$ $Y$ is $(\mathcal{P})$-structural.*

*In particular, the classes of $(\mathcal{P})$-structural and $(\mathcal{P})$-$w^*$-geometrical Banach spaces are closed for quotients (of closed subspaces) and isomorphisms.*

*Proof.* We only need to prove the first implication. Indeed, from $T$ we can construct an onto bounded linear operator $\tilde{T} : X \times \mathbb{R} \to Y \times \mathbb{R}$ given by $\tilde{T}(x,t) = (Tx, t)$. So, assuming that $X$ is $(\mathcal{P})$-structural we would have, by Proposition 2.3.9, that $X \times \mathbb{R}$ is $(\mathcal{P})$-$w^*$-geometrical and therefore, that $Y \times \mathbb{R}$ is also $(\mathcal{P})$-$w^*$-geometrical. Finally, applying Proposition 2.3.9 again, we would get that $Y$ is $(\mathcal{P})$-structural.

Let us prove then the first statement. Assume that $X$ is $(\mathcal{P})$-$w^*$-geometrical and let $K_{Y^*}$ be a $w^*$-closed convex set of $Y$ with $\operatorname{int}(\operatorname{dom}\sigma_{K_{Y^*}}) \neq \emptyset$. We need to prove that $\mathcal{P}[K_{Y^*}] \cap \operatorname{int}(\operatorname{dom}\sigma_{K_{Y^*}})$ is nonempty. Consider the set $K_{X^*} = T^*(K_{Y^*})$, which is a $w^*$-closed convex set of $X^*$. Note that for all $x \in X$,

$$\sigma_{K_{X^*}}(x) = \sup_{y^* \in K_{Y^*}} \langle x, T^* y^* \rangle = \sup_{y^* \in K_{Y^*}} \langle Tx, y^* \rangle = \sigma_{K_{Y^*}}(Tx),$$

which implies that $\operatorname{dom}\sigma_{K_{X^*}} = T^{-1}(\operatorname{dom}\sigma_{K_{Y^*}})$ and, since $T$ is open, it ensues that $T(\operatorname{int}(\operatorname{dom}\sigma_{K_{X^*}})) = \operatorname{int}(\operatorname{dom}\sigma_{K_{Y^*}})$. In particular, $\operatorname{int}(\operatorname{dom}\sigma_{K_{X^*}}) \neq \emptyset$ and therefore there exists $x \in \operatorname{int}(\operatorname{dom}\sigma_{K_{X^*}})$ with $\mathcal{P}(K_{X^*}, x) = 1$. Since $(\mathcal{P})$ is transitive,

$$\mathcal{P}(K_{Y^*}, Tx) = \mathcal{P}(\sigma_{K_{Y^*}} \circ T, x) = \mathcal{P}(\sigma_{K_{X^*}}, x) = \mathcal{P}(K_{X^*}, x),$$

and so we conclude that $Tx \in \mathcal{P}[K_{Y^*}] \cap \operatorname{int}(\operatorname{dom}\sigma_{K_{Y^*}})$, finishing the proof. $\square$

To finish the study of convex smooth-like properties, we would like to make a last reduction: An equivalence between $(\mathcal{P})$-structurality and the nonemptyness of $\mathcal{P}_X[p]$, for each equivalent norm $p$ on $X$. This idea is motivated by characterization $(g)$ of Asplund spaces given in Theorem 2.2.5.

Since this equivalence would imply that $(\mathcal{P})$-structural spaces and $(\mathcal{P})$-$w^*$-geometrical spaces coincide, it is clear that it is a difficult goal. Nevertheless, we present a partial result: In the case that $X$ is a *separable Banach space*, then $(\mathcal{P})$-$w^*$-geometrical spaces can be characterized by the nonemptyness of $\mathcal{P}_X[p]$ for each equivalent norm $p$ on $X$, whenever the convex smooth-like property $(\mathcal{P})$ fulfills an extra condition, that we call *the sum rule.*

**Definition 2.3.11** (Sum rule) *Let $(\mathcal{P})$ be a convex smooth-like property of Banach spaces. We say that $(\mathcal{P})$ has* the sum rule *if for any Banach space $X$ and any pair of functions $f, g \in \Gamma_0(X)$,*

$$\mathcal{P}_X(f+g,\cdot) = \min\{\mathcal{P}_X(f,\cdot), \mathcal{P}_X(g,\cdot)\}. \tag{2.9}$$

Now, to prove our desired reduction, we need first the following fundamental result about open cones in separable Banach spaces:

**Proposition 2.3.12** *Let $X$ be a separable Banach space. For every open cone $C$ in $X$, there exists a countable family of isomorphisms $\{T_n : X \to X \mid n \in \mathbb{N}\}$, such that*

$$\bigcup_{n\in\mathbb{N}} T_n^{-1}(C) \supseteq X \setminus \{0\}.$$

*Proof.* Let $X$ be a separable Banach space and $C$ be an open cone. Fix $x_0 \in \mathbb{S}_X \cap C$ and let $x_0^* \in \mathbb{S}_{X^*}$ such that $\langle x_0^*, x_0\rangle = 1$. Denote $Z_0 := \operatorname{Ker}(x_0^*)$. Since $C$ is open, there exist $\delta > 0$ such that $x_0 + \delta\mathbb{B}_{Z_0} \subseteq C$. Define $C_0 := \overline{\operatorname{cone}}(x_0 + \delta\mathbb{B}_{Z_0})$.

Now, let $\{x_n : n \in \mathbb{N}\}$ a dense set of $\mathbb{S}_X$ and $\{x_n^* : n \in \mathbb{N}\}$ be a subset of $\mathbb{S}_{X^*}$ such that $\langle x_n^*, x_n\rangle = 1$ for all $n \in \mathbb{N}$. Denote $Z_n = \operatorname{Ker}(x_n^*)$. For each $n \in \mathbb{N}$, let us define the set

$$K_n := \overline{\operatorname{cone}}\left(B_X\left(x_n, \tfrac{1}{3}\right)\right) \cap (x_n + Z_n) = \overline{\operatorname{cone}}\left(B_X\left(x_n, \tfrac{1}{3}\right)\right) \cap \{x \in X \ : \ \langle x_n^*, x\rangle = 1\}.$$

Since $\langle x_n^*, x\rangle > 0$ for all $x \in B_X\left[x_n, \frac{1}{3}\right]$ and $\overline{\operatorname{cone}}\left(B_X\left(x_n, \frac{1}{3}\right)\right) \subseteq \operatorname{cone}\left(B_X\left[x_n, \frac{1}{3}\right]\right)$, it is not hard to see that $\langle x_n^*, x\rangle > 0$ for all $x \in \overline{\operatorname{cone}}\left(B_X\left(x_n, \frac{1}{3}\right)\right) \setminus \{0\}$. Thus, we deduce that for each $n \in \mathbb{N}$

$$\overline{\operatorname{cone}}(K_n) = \overline{\operatorname{cone}}\left(B_X\left(x_n, \tfrac{1}{3}\right)\right).$$

Clearly, we also have that $K_n$ is bounded. Indeed, if $x \in K_n$, then there exists $\lambda > 0$ such that $\lambda x \in B_X(x_n, \frac{1}{3})$ and so

$$\lambda = \langle x_n^*, \lambda x\rangle \geq \inf\left\{\langle x_n^*, y\rangle \ : \ y \in B_X\left(x_n, \tfrac{1}{3}\right)\right\} \geq \frac{2}{3}.$$

This yields the inequality

$$\|x\| = \lambda^{-1}\|\lambda x - x_n + x_n\| \leq \tfrac{3}{2}(\|\lambda x - x_n\| + \|x_n\|) \leq 2,$$

proving our claim. Thus, for each $n \in \mathbb{N}$ we can choose $\alpha_n > 0$ such that

$$K_n \subseteq x_n + \alpha_n\mathbb{B}_{Z_n},$$

and define the set $C_n = \overline{\operatorname{cone}}(x_n + \alpha_n\mathbb{B}_{Z_n})$. By construction, we have that

$$\mathbb{S}_X \subseteq \bigcup_{n\in\mathbb{N}} B_X(x_n, \tfrac{1}{3}) \subseteq \bigcup_{n\in\mathbb{N}} C_n \setminus \{0\},$$

and so, since each $C_n$ is a cone, we deduce that $X \setminus \{0\} \subseteq \bigcup_{n\in\mathbb{N}} C_n \setminus \{0\}$. Now, fix $n \in \mathbb{N}$. Since $Z_n$ and $Z_0$ are both hyperplanes, they are isomorphic. Thus, we can choose an isomorphism $S_n : Z_n \to Z_0$ such that

$$S_n(\alpha_n\mathbb{B}_{Z_n}) \subseteq \delta\mathbb{B}_{Z_0}.$$

Define then the isomorphism $T_n : X \to X$ given by $T_n(x_n) = x_0$ and $T_n\big|_{Z_n} = S_n$. It is not hard to see that, by construction

$$T_n(C_n) = \overline{\text{cone}}\, T_n(x_n + \alpha_n \mathbb{B}_{Z_n}) \subseteq \overline{\text{cone}}(x_0 + \delta \mathbb{B}_{Z_0}) = C_0.$$

Repeating this construction for each $n \in \mathbb{N}$ we deduce that

$$X \setminus \{0\} = \bigcup_{n\in\mathbb{N}} C_n \setminus \{0\} \subseteq \bigcup_{n\in\mathbb{N}} T_n^{-1}(C_0 \setminus \{0\}) \subseteq \bigcup_{n\in\mathbb{N}} T_n^{-1}(C),$$

which finishes the proof. □

**Remark 2.3.13** The latter proposition allows us to cover the whole space by "rotating" and "widening" any cone with nonempty interior. This idea is a generalization of the fact that any separable space can be covered (without considering 0, of course) with countable many open half-spaces, which are those induced by any separating countable family of functionals of the dual.

**Theorem 2.3.14** *Let $X$ be a separable Banach space and let $(\mathcal{P})$ be a convex smooth-like property following the sum rule. Then, $X$ is $(\mathcal{P})$-$w^*$-geometrical if and only if for every equivalent norm $p$ on $X$, there exists a nonzero point $x \in X$ such that $\mathcal{P}_X(p, x) = 1$.*

*Proof.* Since the necessity is direct, we only need to prove the sufficiency. Reasoning by contradiction, suppose that $X$ is not $(\mathcal{P})$-$w^*$-geometrical, that is (by Theorem 2.3.7(d)), there exists an equivalent norm $p$ on $X$ and an open subset $U$ of $X$ such that for all $u \in U$, $\mathcal{P}_X(p, u) = 0$.

Consider then the set $C = \text{int}(\text{cone}(U))$, which is an open cone in $X$. By set-consistency of $(\mathcal{P})$ (see Definition 2.3.1(iii)), we know that for every $u \in C$, $\mathcal{P}_X(p, u) = 0$. Also, by Proposition 2.3.12, there exists a countable family $\{T_n \ :\ n \in \mathbb{N}\}$ of linear onto automorphisms, such that

$$\bigcup_{n\in\mathbb{N}} T_n^{-1}(C) \supseteq X \setminus \{0\}.$$

Consider then the function $\rho : X \to \mathbb{R}_+$ given by

$$\rho(x) = \sum_{n=0}^{\infty} 2^{-n} \|T_n\|^{-1} (p \circ T_n)(x),$$

where $T_0 = \text{id}_X$. Clearly, we have that $\rho$ is a norm on $X$. Also, since $p$ is an equivalent norm on $X$, there exist two constants $K_1, K_2 > 0$ such that $p(\cdot) \leq K_1 \|\cdot\|$ and $\|\cdot\| \leq K_2 p(\cdot)$. This entails that

$$p(x) \leq \rho(x) \leq \sum_{n=0}^{\infty} 2^{-n} \|T_n\|^{-1} K_1 \|T_n x\| \leq \sum_{n=0}^{\infty} 2^{-n} K_1 \|x\| \leq 2K_1K_2 p(x).$$

Therefore, $\rho$ is an equivalent norm on $X$. Now, fix $x \in X \setminus \{0\}$. By construction, there exists $n \in \mathbb{N}$ such that $T_n(x) \in C$, and so by transitivity and set-consistency of $(\mathcal{P})$ (see Definition 2.3.1(ii)-(iii)), and using that $p \circ T_n$ is positively homogeneous, we see that

$$\mathcal{P}_X(2^{-n}\|T_n\|^{-1}(p \circ T_n), x) = \mathcal{P}_X(p \circ T_n, x) = \mathcal{P}_X(p, T_n(x)) = 0.$$

Consider then the function $\rho_{-n} : X \to \mathbb{R}_+$ given by $\rho_{-n}(y) = \sum_{k \neq n}^{\infty} 2^{-k}\|T_k\|^{-1}(p \circ T_k)(y)$. Clearly, $\rho_{-n} \in \Gamma_0(X)$ and also $\rho = \rho_{-n} + 2^{-n}\|T_n\|^{-1}(p \circ T_n)$. Finally, by sum rule, we deduce that

$$\mathcal{P}_X(\rho, x) = \min\{\mathcal{P}_X(\rho_{-n}, x), \mathcal{P}_X(2^{-n}\|T_n\|^{-1}(p \circ T_n), x)\} = 0.$$

Since $x$ is arbitrary, we conclude that $\rho$ is an equivalent norm satisfying that $\mathcal{P}_X(\rho, x) = 0$ for all $x \in X \setminus \{0\}$, which contradicts our hypothesis. The proof is then complete. $\square$

### 2.3.2 Convex $w^*$-smooth-like properties

In this subsection, motivated by the notion of $w^*$-Asplund spaces, we will study the case when the smooth-like property holds for the conjugate functions. This case appears when there is a geometric property for the convex sets of the primal. Even though the results and proofs are practically the same as the ones of the latter section, we present this case separately for the sake of order and also since there are some additional delicate details concerning the conjugate notion.

**Definition 2.3.15** ($w^*$-smooth-like properties) *A property $(\mathcal{P})$ of Banach spaces is a* convex $w^*$-smooth-like property *if for each Banach space $X$, $D(X) = \Gamma_0(X^*, w^*) \times X^*$ and the function $\mathcal{P}_X$ satisfy the following conditions:*

*(i) $\mathcal{P}_X$ is **local**: For each pair of two functions $f, g \in \Gamma_0(X^*, w^*)$ and for each open set $U \subseteq X^*$ we have that*

$$f\big|_U = g\big|_U \Rightarrow \mathcal{P}_X(g, \cdot)\big|_U = \mathcal{P}_X(f, \cdot)\big|_U.$$

*(ii) $\mathcal{P}_X$ is $w^*$-**transitive**: For each other Banach space $Y$ such that there exists an one-to-one bounded linear operator $T : Y \to X$ with closed range, we have that for all $f \in \Gamma_0(Y^*, w^*)$*

$$\mathcal{P}_X(f \circ T^*, x^*) = \mathcal{P}_Y(f, T^*x^*),\ \forall x^* \in X^*.$$

*(iii) $\mathcal{P}_X$ is **set-consistent**: For every closed convex set $K$ of $X$ we have that*

*(iii.a) $\forall x^* \in X^*, \forall t > 0$, $\mathcal{P}_X(\sigma_K, x^*) = \mathcal{P}(\sigma_K, tx^*)$.*

*(iii.b) $\forall x \in X$, $\mathcal{P}_X(\sigma_{K+x}, \cdot) = \mathcal{P}_X(\sigma_K, \cdot)$.*

*(iv)* $\mathcal{P}_X$ *is* ***epigraphical****: For each function* $f^* \in \Gamma_0(X^*, w^*)$ *(where* $f$ *is the function in* $\Gamma_0(X)$ *for which* $f^*$ *is its conjugate) and each* $x^* \in X^*$*,*

$$\mathcal{P}_X(f^*, x^*) = \mathcal{P}_{X\times\mathbb{R}}(\sigma_{\operatorname{epi} f}, (x^*, -1)).$$

Again, for reducing notation we will write simply $\mathcal{P}(\cdot,\cdot)$, omitting the index. Also, in the case of a support function $\sigma_K$ (where $K$ is a closed convex set of $X$), we will sometimes write $\mathcal{P}(K,\cdot)$ instead of $\mathcal{P}(\sigma_K,\cdot)$.

Finally, for a function $f \in \Gamma_0(X^*, w^*)$, we will denote by $\mathcal{P}[f] := \{x^* \in \operatorname{dom} f : \mathcal{P}(f, x^*) = 1\}$. Consistently, if $f = \sigma_K$, then we will write sometimes $\mathcal{P}[K]$ instead of $\mathcal{P}[\sigma_K]$.

**Definition 2.3.16** (($\mathcal{P}$)-$w^*$-structural spaces) *Let* ($\mathcal{P}$) *be a convex smooth-like property. A Banach space* $X$ *is said to be*

1. ($\mathcal{P}$)***-$w^*$-structural*** *if for each* $f \in \Gamma_0(X^*, w^*)$*,* $\mathcal{P}[f]$ *is dense in* $\operatorname{int}(\operatorname{dom} f)$*.*

2. ($\mathcal{P}$)***-geometrical*** *if for each closed convex bounded set* $K \subseteq X$*,* $\mathcal{P}[K]$ *is dense in* $X^*$*.*

**Lemma 2.3.17** *Let* $X$ *be a Banach space,* $f \in \Gamma_0(X^*, w^*)$ *and* $x_0^* \in \operatorname{int}(\operatorname{dom} f)$*. Then, there exist an open neighborhood* $U \in \mathcal{N}_{X^*}(x_0^*)$ *and a convex Lipschitz-continuous function* $\tilde{f} \in \Gamma_0(X^*, w^*)$ *such that*

$$f\big|_U = \tilde{f}\big|_U.$$

*Proof.* As in proof of Lemma 2.3.3, we know that there exists $M > 0$ large enough and $U \in \mathcal{N}_{X^*}(x_0^*)$ such that the function

$$\tilde{f} := f\square(M\|\cdot\|_*)$$

is convex, $M$-Lipschitz and it coincides with $f$ in $U$. Thus, we only need to prove that $\tilde{f}$ is $w^*$-lower semicontinuous. Denote $g := f^*\big|_X$, that is, the unique function in $\Gamma_0(X)$ such that $g^* = f$. Since $M$ can be considered arbitrarily large, we may assume that

$$\big(\operatorname{int}(\operatorname{dom} I_{M\mathbb{B}_X})\big) \cap \operatorname{dom} g = B_X(0, M) \cap \operatorname{dom} g \neq \emptyset,$$

and so, the function $I_{M\mathbb{B}_X}$ is continuous at some point of $\operatorname{dom}(I_{M\mathbb{B}_X}) \cap \operatorname{dom} g$. By Proposition 1.2.11(c), we deduce

$$(I_{M\mathbb{B}_X} + g)^* = (I_{M\mathbb{B}_X})^*\square g^* = \tilde{f},$$

where the second equality follows from the straight forward relation $(I_{M\mathbb{B}_X})^* = M\|\cdot\|_*$. This yields the inclusion $\tilde{f} \in \Gamma_0(X^*, w^*)$, finishing the proof. $\square$

**Proposition 2.3.18** *The following assertions are equivalent:*

(a) *For each* $f \in \Gamma_0(X^*, w^*)$ *with* $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$, $\mathcal{P}[f] \cap \operatorname{int}(\operatorname{dom} f) \neq \emptyset$.

(b) $X$ *is* $(\mathcal{P})$-$w^*$*-structural.*

(c) *For each* $w^*$*-lsc continuous proper convex function* $f : X^* \to \mathbb{R}_\infty$, $\mathcal{P}[f]$ *is dense in* $\operatorname{int}(\operatorname{dom} f)$.

(d) *For each real-valued function* $f \in \Gamma_0(X^*, w^*)$, $\mathcal{P}[f]$ *is dense in* $X$.

*Proof.* The implications $(b) \Rightarrow (c) \Rightarrow (d)$ are obvious. The implication $(a) \Rightarrow (b)$ is analogous to the proof of the implication $(a) \Rightarrow (b)$ in Proposition 2.3.4, using the fact that if $B \subseteq X^*$ is a $w^*$-closed convex set then $I_B \in \Gamma_0(X^*, w^*)$, and also the fact that for any two functions $f, g \in \Gamma_0(X^*, w^*)$, the sum $f + g$ also belongs to $\Gamma_0(X^*, w^*)$.

To prove the implication $(d) \Rightarrow (a)$, assume that $(d)$ holds and let $f \in \Gamma_0(X^*, w^*)$ with $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$. Choose any $x^* \in \operatorname{int}(\operatorname{dom} f)$. Lemma 2.3.17 ensues that there exist an open neighborhood $U \in \mathcal{N}_{X^*}(x^*)$ and a Lipschitz-continuous function $\tilde{f} \in \Gamma_0(X^*, w^*)$ such that $f\big|_U = \tilde{f}\big|_U$. The rest of proof follows exactly as in Proposition 2.3.4. □

**Lemma 2.3.19** *Let* $K$ *be a closed convex set of* $X$ *with* $\operatorname{int}(\operatorname{dom} \sigma_K) \neq \emptyset$ *and* $x^* \in \operatorname{int}(\operatorname{dom} \sigma_K)$ *with* $x^* \neq 0$. *Then, there exists an equivalent norm* $p$ *on* $X$, *a point* $x_0 \in X$ *and an open neighborhood* $U$ *of* $x^*$ *such that*

$$\sigma_{K+x_0}(u^*) = p_*(u^*), \ \forall u^* \in U.$$

*Proof.* The proof is exactly the same as the proof of Lemma 2.3.6, considering for the case $\eta = \sigma_K(x^*) > 0$ the following sets: $K_1 = X \cap \partial_{\eta/2}\sigma_K(x^*)$, $K_2 = \overline{\text{co}}[K_1 \cup (-K_1)]$, $B = \frac{\eta}{2\|x^*\|_*}\mathbb{B}_X$ and $K_3 = \overline{\text{co}}[B \cup K_2]$. □

**Theorem 2.3.20** *The following assertions are equivalent:*

(a) *For each closed convex set* $K \subseteq X$ *with* $\operatorname{int}(\operatorname{dom} \sigma_K) \neq \emptyset$, $\mathcal{P}[\sigma_K] \cap \operatorname{int}(\operatorname{dom} \sigma_K) \neq \emptyset$.

(b) *For each closed convex set* $K \subseteq X$, $\mathcal{P}[\sigma_K]$ *is dense in* $\operatorname{int}(\operatorname{dom} \sigma_K)$.

(c) $X$ *is* $(\mathcal{P})$*-geometrical.*

(d) *For each equivalent norm* $p$ *on* $X$, $\mathcal{P}[p]$ *is dense in* $X^*$.

*Proof.* The proofs of the equivalences are analogous to the proofs of the equivalences in Theorem 2.3.7, using Proposition 1.2.26 for the locally convex topology $\theta = w^*$ on $X^*$, and noting that for any $w^*$-compact convex set $B$ of $X^*$ not containing 0, the set

$$C = \operatorname{cone}(B)$$

is a $w^*$-closed convex cone. □

**Proposition 2.3.21** *A Banach space $X$ is $(\mathcal{P})$-$w^*$-structural if and only if $X \times \mathbb{R}$ is $(\mathcal{P})$-geometrical.*

*Proof.* Using Lemma 2.3.8 with $\theta = w^*$ in $X^*$, we can prove, analogously as in proof of Proposition 2.3.9, the sufficiency of the equivalence.

Let us prove the necessity. Assume that $X$ is $(\mathcal{P})$-$w^*$-structural and fix an equivalent norm $p$ on $X \times \mathbb{R}$. As in Proposition 2.3.9, we can define the function $\varphi : X \to \mathbb{R}$ given by $\varphi(x) := \inf\{t \ : \ p(x,t) \leq 1\}$, and derive that $\operatorname{epi}\varphi = \mathbb{B}_{(X,p)} + L$, where $L = \{(0,t) \ : \ t \geq 0\}$. This yields that $\varphi \in \Gamma_0(X)$. Noting also that

$$\sigma_{\operatorname{epi}\varphi}\big|_{X^*\times]-\infty,0[} = \sigma_{B_{(X\times\mathbb{R},p)}}\big|_{X^*\times]-\infty,0[} = p_*\big|_{X^*\times]-\infty,0[}$$

we get by Lemma 2.3.8 (applied with $\theta = w^*$ in $X^*$) that $\operatorname{int}(\operatorname{dom}\varphi) \neq \emptyset$, and hence $\mathcal{P}[\varphi]$ is dense in $\operatorname{int}(\operatorname{dom}\varphi)$ by the $(\mathcal{P})$-$w^*$-structural property of $X$. Using that $\mathcal{P}$ is epigraphical and set consistent, we deduce that $\mathcal{P}[\operatorname{epi}\varphi]$ is dense in $\operatorname{int}(\operatorname{dom}\sigma_{\operatorname{epi}\varphi}) = X^*\times]-\infty,0[$. Again, we conclude that $\mathcal{P}[p]$ is dense in $X^*\times]-\infty,0[$, using the fact that

and that $(\mathcal{P})$ is local. Now, consider the isomorphism $T : X \times \mathbb{R} \to X \times \mathbb{R}$ given by $T(x,t) = (x,-t)$. It is not hard to realize that $T^*(x^*,s) = (x^*,-s)$ for all $(x^*,s) \in X \times \mathbb{R}$. Since $q_* := p^* \circ T^*$ is an equivalent dual norm on $X^* \times \mathbb{R}$, we get by the same arguments as above that $\mathcal{P}[q_*]$ is dense in $X^*\times]-\infty,0[$. Finally, since $T$ is one-to-one and of closed range, we can use the $w^*$-transitivity of $(\mathcal{P})$ to deduce that

$$\forall (x^*,s) \in X^* \times \mathbb{R},\ \mathcal{P}(p_*, T^*(x^*,s)) = \mathcal{P}(p_* \circ T^*, (x^*,s)) = \mathcal{P}(q_*, (x^*,t)).$$

Noting that $T^*\left(X^*\times]0,+\infty[\right) = X^*\times]-\infty,0[$, we conclude that $\mathcal{P}[p_*]$ is also dense in $X^*\times]0,+\infty[$. Since $p$ is arbitrary, the conclusion follows by Proposition 2.3.18. $\square$

**Proposition 2.3.22** *Let $X, Y$ be two Banach spaces with an one-to-one bounded linear operator $T : Y \to X$ with closed range. Then,*

*(a) $X$ is $(\mathcal{P})$-geometrical $\Rightarrow$ $Y$ is $(\mathcal{P})$-geometrical.*

*(b) $X$ is $(\mathcal{P})$-$w^*$-structural $\Rightarrow$ $Y$ is $(\mathcal{P})$-$w^*$-structural.*

*In particular, the classes of $(\mathcal{P})$-$w^*$-structural and $(\mathcal{P})$-geometrical are closed for closed subspaces and isomorphisms.*

*Proof.* As in proof of Proposition 2.3.10, we only need to prove the first implication. Assume then that $X$ is $(\mathcal{P})$-geometrical and let $K_Y$ be a closed convex set of $Y$ with $\operatorname{int}(\operatorname{dom}\sigma_{K_Y}) \neq \emptyset$. Let $K_X = T(K_Y)$, which is a closed convex set of $X$, since $T$ is one-to-one with closed range. Now, for each $x^* \in X^*$,

$$\sigma_{K_Y}(T^*x^*) = \sup_{y\in K_Y} \langle T^*x^*, y\rangle = \sup_{y\in K_Y} \langle x^*, Ty\rangle = \sigma_{K_X}(x^*),$$

and so, $\operatorname{dom}\sigma_{K_X} = (T^*)^{-1}(\operatorname{dom}\sigma_{K_Y})$ and $T^*(\operatorname{int}(\operatorname{dom}\sigma_{K_X})) = \operatorname{int}(\operatorname{dom}\sigma_{K_Y})$, according to the fact that $T^*$ is an open map. The rest of the proof follows exactly as the proof of Proposition 2.3.10, using $w^*$-transitivity instead of transitivity. □

We finish this subsection by presenting the dual version of the reduction given in Theorem 2.3.14 in the separable case. It is worth pointing out that here the hypothesis will be the separability of the primal space with a possible nonseparable dual. In this case, we still can provide a covering of the dual space by "rotating" and "widening" countable many times any open cone. Nevertheless, the proof is more delicate. In fact, as the reader can see, Proposition 2.3.24 below requires some additional arguments, since the isomorphisms involved in the rotations are also adjoint operators.

**Definition 2.3.23** ($w^*$-Sum rule) *Let $(\mathcal{P})$ be a convex $w^*$-smooth-like property of Banach spaces. We say that $(\mathcal{P})$ has* the $w^*$-sum rule *if for any Banach space $X$ and any pair of functions $f, g \in \Gamma_0(X^*, w^*)$,*

$$\mathcal{P}_X(f+g,\cdot) = \min\{\mathcal{P}_X(f,\cdot), \mathcal{P}_X(g,\cdot)\}. \tag{2.10}$$

**Proposition 2.3.24** *Let $X$ be a separable Banach space. For every open cone $C$ of the dual space $X^*$, there exists a countable family of isomorphisms $\{T_n : X \to X \ : \ n \in \mathbb{N}\}$ such that*

$$\bigcup_{n\in\mathbb{N}} (T_n^*)^{-1}(C) \supseteq X^* \setminus \{0\}.$$

*Proof.* Let $C$ be an open cone of $X^*$ and choose $x_0^* \in \mathbb{S}_{X^*} \cap C$ such that $x_0^*$ attains its norm at some point $x_0 \in \mathbb{S}_X$. Choose $\delta \in ]0, 1/3[$ small enough such that $B_{X^*}[x_0^*, \delta] \subseteq C$ and define $C_0 := \operatorname{cone}(B_{X^*}[x_0^*, \delta])$. Since $B_{X^*}[x_0^*, \delta]$ is $w^*$-compact and does not contains zero, we have that $C_0$ is $w^*$-closed, and so, defining $C_{X,0} := X \cap (C_0)^o$, we get that $(C_{X,0})^o = C_0$. Also, noting that $C_0 \setminus \{0\} = ]0, +\infty[ \cdot B_{X^*}[x_0^*, \delta]$, we deduce that $C_0 \setminus \{0\} \subseteq C$.

Let $Z_0 := \operatorname{Ker}(x_0^*)$ and consider the set $K = C_{X,0} \cap (Z_0 + x_0)$. We claim that $K$ is bounded: Indeed, if $K$ is not bounded, there exists a direction $z \in Z_0$ such that

$$\{x_0 + tz \ : \ t > 0\} \subseteq K.$$

Fix $x^* \in X^*$ such that $\langle x^*, z\rangle = 1$. Since $x_0 \in \operatorname{int}(C_0)$, there exists $\lambda > 0$ small enough such that $x_0^* + \lambda x^* \in \operatorname{int}(C_0)$. Thus, for each $t > 0$ we would have

$$\langle \lambda x^* + x_0^*, x_0 + tz\rangle = \lambda t + \langle \lambda x^* + x_0^*, x_0 + tz\rangle < 1,$$

which clearly is not possible. Thus, $K$ has to be bounded as we claimed. Therefore, there exists $\alpha > 0$ such that $\alpha\mathbb{B}_{Z_0} \supseteq K$. Also, as we did in proof of Proposition 2.3.12, we have that $C_{X,0} = \overline{\operatorname{cone}}(K)$. Define the closed cone $D_{X,0} := \overline{\operatorname{cone}}(x_0 + \alpha\mathbb{B}_{Z_0})$ and $D_0 := (D_{X,0})^o$. Since $C_{X,0} = \overline{\operatorname{cone}}(K) \subseteq D_{X,0}$, we deduce that $D_0 \subseteq C_0$.

Now, consider a dense subset $\{x_n : n \in \mathbb{N}\}$ of $\mathbb{S}_X$, and a subset $\{x_n^* : n \in \mathbb{N}\}$ of $\mathbb{S}_{X^*}$ such that $\langle x_n^*, x_n \rangle = 1$. For each $n \in \mathbb{N}$, we will denote $Z_n = \mathrm{Ker}(x_n^*)$. For each $n \in \mathbb{N}$, it is clear that $x_n + \frac{1}{4}\mathbb{B}_{Z_n} \subseteq B_X\left(x_n, \frac{1}{3}\right)$, and consider the sets $C_{X,n} = \overline{\mathrm{cone}}(B_X\left(x_n, \frac{1}{3}\right))$ and $D_{X,n} = \overline{\mathrm{cone}}\left(x_n + \frac{1}{4}\mathbb{B}_{Z_n}\right)$. Denoting $C_n := (C_{X,n})^o$ and $D_n := (D_{X,n})^o$, it is not hard to see that $C_n \subseteq D_n$, since $D_{X,n} \subseteq C_{X,n}$ by construction.

We claim that $\bigcup C_n = X^* \setminus \{0\}$. Indeed, fix $x^* \in \mathbb{S}_{X^*}$ and choose $n \in \mathbb{N}$ such that $\langle x^*, x_n \rangle < -\frac{2}{3}$. Thus, for every $x \in B_X\left(x_n, \frac{1}{3}\right)$ we have that

$$\langle x^*, x \rangle = \langle x^*, x - x_n \rangle + \langle x^*, x_n \rangle < \|x - x_n\| - \tfrac{2}{3} \leq -\tfrac{1}{3}.$$

Thus, $x^* \in \left[\overline{\mathrm{cone}}\left(B_X\left(x_n, \frac{1}{3}\right)\right)\right]^o = C_n$. By arbitrariness of $x^*$, we deduce that $\mathbb{S}_{X^*} \subseteq \bigcup C_n$, which proves our claim, since the sets $C_n$ are cones.

Since $Z_0$ and $Z_n$ are both closed hyperplanes of $X$, they are isomorphic. Thus, we can choose an isomorphism $S_n : Z_0 \to Z_n$ such that

$$S_n(\alpha \mathbb{B}_{Z_0}) \subseteq \tfrac{1}{4}\mathbb{B}_{Z_n}.$$

Define then the isomorphism $T_n : X \to X$ given by $T_n(x_0) = x_n$ and $T_n\big|_{Z_0} = S_n$. It is not hard to see that, by construction

$$T_n(C_{X,0}) \subseteq T_n(D_{X,0}) = \overline{\mathrm{cone}}(T_n(x_0 + \alpha\mathbb{B}_{Z_0})) \subseteq \overline{\mathrm{cone}}\left(x_n + \tfrac{1}{4}\mathbb{B}_{Z_n}\right) = D_{X,n},$$

which entails the inclusion $D_n = (D_{X,n})^o \subseteq (T_n(C_{X,0}))^o = (T_n^*)^{-1}(C_0)$. Finally, we can write

$$X^* \setminus \{0\} = \bigcup_{n\in\mathbb{N}} C_n \setminus \{0\} \subseteq \bigcup_{n\in\mathbb{N}} D_n \setminus \{0\} \subseteq \bigcup_{n\in\mathbb{N}} (T_n^*)^{-1}(C_0 \setminus \{0\}) \subseteq \bigcup_{n\in\mathbb{N}} (T_n^*)^{-1}(C),$$

which finishes the proof. □

**Theorem 2.3.25** *Let $X$ be a separable Banach space, and let $(\mathcal{P})$ be a convex $w^*$-smooth-like property fulfilling the $w^*$-sum rule. Then, $X$ is $(\mathcal{P})$-geometrical if and only if for every equivalent norm $p$ on $X$, there exists a nonzero linear functional $x^* \in X^*$ such that $\mathcal{P}_X(p_*, x^*) = 1$.*

*Proof.* As in the proof of Theorem 2.3.14, let us suppose that there exists an equivalent norm $p$ on $X$ and an open set $U \subseteq X^*$ such that $\mathcal{P}_X(p_*, u^*) = 0$, for all $u \in U$. By set consistency of $(\mathcal{P})$, we know that $\mathcal{P}_X(p_*, x^*) = 0$ for all $x^* \in C := ]0, +\infty[\cdot U$, where the latter set is an open cone.

Using Proposition 2.3.24, there exists a countable family of isomorphisms $\{T_n : X \to X \mid n \in \mathbb{N}\}$ such that

$$X^* \setminus \{0\} \subseteq \bigcup_{n\in\mathbb{N}} (T_n^*)^{-1}(C).$$

Consider then the function $\rho_* : X^* \to \mathbb{R}_+$ given by

$$\rho_*(x^*) := \sum_{n=0}^{\infty} 2^{-n} \|T_n^*\|^{-1} (p_* \circ T_n^*)(x^*),$$

where $T_0^* = \mathrm{id}_{X^*}$, and the operators norm $\|\cdot\|$ in $\mathcal{L}(X^*; X^*)$ is taken with respect to the norm $p_*$. Noting that for every $x^* \in X^*$ we can write

$$p_*(x^*) \leq \rho_*(x^*) \leq \sum_{n=0}^{\infty} 2^{-n} p_*(x^*) = 2p_*(x^*),$$

we have that $\rho_*$ is an equivalent norm on $X^*$. To prove then that it is an equivalent dual norm, it is enough to show that $\rho_*$ is $w^*$-lower semicontinuous. To do so, according to the sublinearity of $\rho_*$ and Propositions 1.2.26 and 1.2.27, it is enough to show that $\mathbb{B}_{(X^*,\rho_*)} = \{x^* \in X^* \ : \ \rho_*(x^*) \leq 1\}$ is $w^*$-closed.

Let $(x_\lambda^*)_{\lambda\in\Lambda}$ be a net in $\mathbb{B}_{(X^*,\rho_*)}$ $w^*$-converging to some point $x^* \in X^*$ and let $\varepsilon > 0$. Since $\rho_* \leq \sum_{n=0}^{\infty} 2^{-n} p_*$ and recalling that $\mathbb{B}_{(X^*,\rho_*)} \subseteq 2\mathbb{B}_{(X^*,p_*)}$, there exists $n \in \mathbb{N}$ such that

$$\sum_{k>n}^{\infty} 2^{-k} \|T_k^*\|^{-1} (p_* \circ T_k^*)(y^*) \leq \varepsilon, \quad \forall y^* \in \mathrm{cl}_{w^*}(\mathbb{B}_{(X^*,\rho_*)}).$$

In particular, this inequality holds for every $x_\lambda^*$ and for the limit point $x^*$. Defining $\rho_{*,n} := \sum_{k=0}^{n} 2^{-k} \|T_k^*\|^{-1} (p_* \circ T_k^*)$, we have that $\rho_{*,n} \in \Gamma_0(X^*, w^*)$ (since it is a finite sum of $w^*$-lsc functions) and so, its level sets are $w^*$-closed. Thus, since $\rho_{*,n}(x_\lambda^*) \leq \rho_*(x_{\lambda^*}) \leq 1$ for all $\lambda \in \Lambda$ and $x_\lambda^* \rightharpoonup^{w^*} x^*$, we get that $\rho_{*,n}(x^*) \leq 1$. Finally, we deduce that

$$\rho_*(x^*) \leq \rho_{*,n}(x^*) + \varepsilon \leq 1 + \varepsilon.$$

Since $\varepsilon$ is arbitrary, we deduce that $\rho_*(x^*) \leq 1$, which proves that $\mathbb{B}_{(X^*,\rho_*)}$ is $w^*$-closed, entailing the $w^*$-lower semicontinuity of $\rho_*$.

The rest of the proof follows exactly as in Theorem 2.3.14, using the $w^*$-transitivity of $(\mathcal{P})$ and noting that, by the same reasoning as above, for each $n \in \mathbb{N}$, the function $\rho_{*,-n} := \sum_{k\neq n} 2^{-k} \|T_k^*\|^{-1} (p_* \circ T_k^*)$ belongs to $\Gamma_0(X^*, w^*)$ as well. □

## 2.4 Some Examples and Final comments

As we have shown, convex smooth-like and $w^*$-smooth-like properties have remarkable stability behaviors which rely only on very natural characteristics. Even though they are motivated by Asplund and $w^*$-Asplund spaces (and therefore, by Fréchet-differentiability), these are not the only examples of such a type of properties.

Another family of spaces that can be fitted in this framework are Gâteaux Differentiable spaces (GDS): A Banach space is said to be GDS, if for every function $f \in \Gamma_0(X)$ with $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$, the set

$$\{x \in \operatorname{int}(\operatorname{dom} f) \ : \ f \text{ is G-differentiable at } x\}$$

is dense in $\operatorname{int}(\operatorname{dom} f)$. It is intuitive that defining the property $(\mathcal{P})$ by setting $D(X) = \Gamma_0(X) \times X$ and

$$\mathcal{P}_X(f, x) = 1 \iff f \text{ is G-differentiable at } x,$$

then $(\mathcal{P})$ is a convex smooth-like property and the family GD spaces are the $(\mathcal{P})$-structural spaces. These family of spaces is beautifully developed in [36, Ch. 6]. As final comments of this chapter, we consider *$\beta$-differentiability spaces*, where $\beta$ is any bornology of Banach spaces. The differentiability with respect to a bornology can be tracked back to the seminal work of J. Borwein and D. Preiss [4], and it is nicely developed in [36, Ch. 4 and Ch. 6]. We will prove that these spaces are in fact $(\beta\mathcal{P})$-structural spaces for some convex smooth-like property $(\beta\mathcal{P})$, associated to the bornology $\beta$.

**Definition 2.4.1** (Bornology) *Let $X$ be a Banach space. A* bornology $\beta$ *on $X$ is a family of* bounded subsets *of $X$ such that*

*(a) For each $\lambda \in \mathbb{R}$ and each $B \in \beta$, $\lambda B \in \beta$.*

*(b) For each $B \in \beta$ and each $B' \subseteq B$, we have that $B' \in \beta$.*

*(c) For each pair $B_1, B_2 \in \beta$, we have that $B_1 \cup B_2 \in \beta$.*

*(d) $\beta$ is a cover of $X$, that is, $X = \bigcup_{B\in\beta} B$.*

There are alternative notions of bornologies, which do not ask stability by scalar multiplication, nor boundedness. Nevertheless, to our purpose this notion is the most suitable one.

**Definition 2.4.2** ($\beta$-Differentiability) *Let $X$ be a Banach space and $\beta$ be a bornology on $X$. We say that a function $f : X \to \mathbb{R}_\infty$ is $\beta$-differentiable at a point $x \in \operatorname{dom} f$ if it is Gâteaux-differentiable at $x$ and the limit defining $\nabla f(x)$ exists $\beta$-uniformly, that is,*

$$\sup_{h\in B} \left| \frac{f(x+th) - f(x)}{t} - \nabla f(x)h \right| \xrightarrow{t\searrow 0} 0, \quad \forall B \in \beta.$$

**Definition 2.4.3** (General Banach Bornology) *A* general Banach bornology $\beta := \{\beta(X)\}$ *is a family of collections of sets indexed by the class of Banach spaces, such that $\beta(X)$ is a bornology on every Banach space $X$ and such that*

*(a) For every two Banach spaces $X$ and $Y$ and any continuous surjective linear operator $T : X \to Y$, we have $T(\beta(X)) = \beta(Y)$.*

(b) *For every Banach space $X$ and every closed subspace $Z$ of $X$, we have that $\beta(Z) \subseteq \beta(X)$.*

General bornologies are in fact the ones that are used in the classic literature. For example, we have the Gâteaux-bornology given by the collection of all finite subsets of $X$, the Hadamard-bornology given by the precompact subsets of $X$, and the Fréchet-bornology given by all bounded sets of $X$.

**Definition 2.4.4** (($\beta\mathcal{P}$) property) *Let $\beta$ be a general Banach bornology. We define the Banach space property $(\beta\mathcal{P})$ as follows: For each Banach space $X$, $D(X) := \Gamma_0(X) \times X$ and the function $\beta\mathcal{P}_X$ is given by*

$$\beta\mathcal{P}_X(f,x) = 1 \iff x \in \operatorname{int}(\operatorname{dom} f) \text{ and } f \text{ is } \beta(X)\text{-differentiable at } x.$$

**Theorem 2.4.5** *For every general Banach bornology $\beta$, the property $(\beta\mathcal{P})$ is a convex smooth-like property fulfilling the sum rule.*

*Proof.* It is direct that $(\beta\mathcal{P})$ is local. Let us prove that it is also transitive. Let $X$ and $Y$ be two Banach spaces, $T : X \to Y$ be a continuous surjective linear operator, and $f \in \Gamma_0(Y)$ such that $\operatorname{int}(\operatorname{dom} f)$ is nonempty. Denoting $g := f \circ T$, it is clear that $\operatorname{int}(\operatorname{dom} g) = T^{-1}(\operatorname{int}(\operatorname{dom} f))$, and so, for every $x \notin \operatorname{int}(\operatorname{dom} g)$,

$$\beta\mathcal{P}_X(g,x) = 0 = \beta\mathcal{P}_Y(g,Tx).$$

Applying a very simple chain rule, we also know that

$$\begin{gathered} x \in \operatorname{int}(\operatorname{dom} g) \text{ and } g \text{ is G-differentiable at } x \\ \Leftrightarrow \\ Tx \in \operatorname{int}(\operatorname{dom} f) \text{ and } f \text{ is G-differentiable at } Tx, \end{gathered} \tag{2.11}$$

and in such a case, $\nabla g(x) = T^*(\nabla f(Tx)) = \nabla f(Tx) \circ T$. Now, let us fix $x \in \operatorname{int}\operatorname{dom} g$ such that $g$ is G-differentiable at $x$. Then, for each $B \in \beta(X)$ we can write

$$\begin{aligned} \sup_{h\in B}\left|\frac{g(x+th)-g(x)}{t} - \nabla g(x)h\right| &= \sup_{h\in B}\left|\frac{f(Tx+tTh)-f(Tx)}{t} - \nabla f(Tx)Th\right| \\ &= \sup_{h\in T(B)}\left|\frac{f(Tx+th)-f(Tx)}{t} - \nabla f(Tx)h\right|. \end{aligned}$$

Then, using that $\beta(Y) = T(\beta(X))$, we conclude by the preceding equality and by equivalence (2.11), that

$$\begin{gathered} x \in \operatorname{int}(\operatorname{dom} g) \text{ and } g \text{ is } \beta(X)\text{-Diff. at } x \\ \Leftrightarrow \\ Tx \in \operatorname{int}(\operatorname{dom} f) \text{ and } f \text{ is } \beta(Y)\text{-Diff. at } Tx, \end{gathered}$$

which entails the transitivity of $(\beta\mathcal{P})$. Let us continue with set-consistency. Fix a nonempty $w^*$-closed convex subset $K$ of $X^*$ and a point $x^* \in X^*$. The sublinearity of $\sigma_K$ yields directly the condition $(iii.a)$ of Definition 2.3.1. Also, since we can write

$$\sigma_{K+x^*}(x) = \sigma_K(x) + \langle x^*, x\rangle \qquad \text{and} \qquad \partial\sigma_{K+x^*}(x) = \partial\sigma_K(x) + x^*,$$

for all $x \in X$, we easily deduce that $\operatorname{int}(\operatorname{dom}\sigma_{K+x^*}) = \operatorname{int}(\operatorname{dom}\sigma_K)$, that the points of Gâteaux-differentiability of $\sigma_{K+x^*}$ and $\sigma_K$ coincide and that if $x \in \operatorname{int}(\operatorname{dom}\sigma_K)$ is such a point, then $\nabla\sigma_{K+x^*}(x) = \nabla\sigma_K(x) + x^*$. Then, we deduce that for any $B \in \beta(X)$

$$\sup_{h\in B}\left|\frac{\sigma_{K+x^*}(x+th) - \sigma_{K+x^*}(x)}{t} - \nabla\sigma_{K+x^*}(x)h\right| = \sup_{h\in B}\left|\frac{\sigma_K(x+th) - \sigma_K(x)}{t} - \nabla\sigma_K(x)h\right|,$$

which combined with the above observations, yields condition $(iii.b)$ of Definition 2.3.1, proving the set-consistency of $(\beta\mathcal{P})$. To conclude that $(\beta\mathcal{P})$ is a convex smooth-like property, it only rests to prove that it is epigraphical. Fix then $f \in \Gamma_0(X)$ and recall by Lemma 2.3.8 that

$$\operatorname{int}(\operatorname{dom}\sigma_{\operatorname{epi} f^*}) = ]0, +\infty[\cdot(\operatorname{int}(\operatorname{dom} f) \times \{-1\}).$$

So, for each $x \notin \operatorname{int}(\operatorname{dom} f)$ we already have that $\beta\mathcal{P}_X(f, x) = \beta\mathcal{P}_{X\times\mathbb{R}}(\sigma_{\operatorname{epi} f^*}, (x, -1))$. Now, assume that $x \in \operatorname{int}(\operatorname{dom} f)$. Since

$$\begin{aligned}\partial f(x) &= \{x^* \in X^* \ : \ f(x) + f^*(x^*) = \langle x^*, x\rangle\}\\ &= \{x^* \in X^* \ : \ \sigma_{\operatorname{epi} f^*}(x, -1) = \langle x^*, x\rangle - f^*(x^*)\}\\ &= \{x^* \in X^* \ : \ (x^*, f^*(x^*)) \in \partial\sigma_{\operatorname{epi} f^*}(x, -1)\}\end{aligned}$$

It is easy to see that $f$ is G-differentiable at $x$ if and only if $\sigma_{\operatorname{epi} f^*}$ is G-differentiable at $(x, -1)$. Thus, we only need to prove that, under the assumption of Gâteaux-differentiability

$$\beta\mathcal{P}_X(f, x) = 1 \iff \beta\mathcal{P}_{X\times\mathbb{R}}(\sigma_{\operatorname{epi} f^*}, (x, -1)) = 1. \tag{2.12}$$

Suppose first that $\sigma_{\operatorname{epi} f^*}$ is $\beta(X\times\mathbb{R})$-differentiable at $(x, -1)$ . Fix $B \in \beta(X)$ and consider $\delta > 0$ small enough such that for every $t \in ]0, \delta[$, $x + th \in \operatorname{int}(\operatorname{dom} f)$ (such a $\delta$ exists, given that $B$ is bounded). Since $\partial f(x+th) \neq \emptyset$ for each $h \in B$ and each $t \in ]0, \delta[$, we get that $f(x+th) = \sigma_{\operatorname{epi} f^*}(x+th, -1)$ and so, denoting $\sigma := \sigma_{\operatorname{epi} f^*}$, for each $t \in ]0, \delta[$ we can write

$$\sup_{h\in B}\left|\frac{f(x+th) - f(x)}{t} - \nabla f(x)h\right| = \sup_{h\in B}\left|\frac{\sigma(x+th, -1) - \sigma(x, -1)}{t} - \nabla\sigma(x, -1)(h, 0)\right|,$$

which entails the $\beta(X)$-differentiability of $f$ at $x$, since the term of the right-hand side converges to 0 as $t \searrow 0$. This proves the sufficiency in the equivalence (2.12). Now, suppose that $f$ is $\beta(X)$-differentiable at $x$, and fix $B \in \beta(X \times \mathbb{R})$. Now for each $(h,s) \in B$ and each $t > 0$, denote $\lambda := |1-ts|^{-1}t$. It is not hard to deduce that for $t > 0$ small enough

$$\begin{aligned}\frac{\sigma(x+th, ts-1) - \sigma(x,-1)}{t} &= \frac{|1-ts|\sigma(x+\lambda(h+sx),-1) - \sigma(x,-1)}{t} \\ &= \frac{f(x+\lambda(h+sx)) - f(x)}{\lambda} + \frac{|ts-1|-1}{t} f(x) \\ &= \frac{f(x+\lambda(h+sx)) - f(x)}{\lambda} - sf(x).\end{aligned}$$

The latter equality tells us

$$\nabla\sigma(x,-1)(h,s) = \nabla f(x)(h+sx) - sf(x).$$

Finally, define the operator $T : X \times \mathbb{R} \to X$ given by $T(h,s) = h + sx$. Clearly $T$ is linear, continuous and surjective. Thus, $\beta(X) = T(\beta(X \times \mathbb{R}))$. Combining these three elements, we get that for $t > 0$ small enough

$$\begin{aligned}&\sup_{(h,s)\in B} \left| \frac{\sigma(x+th, ts-1) - \sigma(x,-1)}{t} - \nabla\sigma(x,-1)(h,s) \right| \\ &= \sup_{(h,s)\in B} \left| \frac{f(x+\lambda(h+sx)) - f(x)}{\lambda} - \nabla f(x)(h+sx) \right| \\ &= \sup_{h'\in T(B)} \left| \frac{f(x+\lambda h') - f(x)}{\lambda} - \nabla f(x)h' \right|.\end{aligned}$$

Noting that $\lambda \searrow 0$ as $t \searrow 0$, we deduce that the last term of this equality converges to 0 as $t \searrow 0$ (provided $T(B) \in \beta(X)$), which entails that

$$\lim_{t\searrow 0} \sup_{(h,s)\in B} \left| \frac{\sigma(x+th, ts-1) - \sigma(x,-1)}{t} - \nabla\sigma(x,-1)(h,s) \right| = 0.$$

This proves that $\sigma$ is $\beta(X \times \mathbb{R})$-differentiable at $(x,-1)$, which proves the necessity in the equivalence 2.12. Thus, $(\beta\mathcal{P})$ is epigraphical, which proves that it is a convex smooth-like property.

Now, let us check that $(\beta\mathcal{P})$ fulfills the sum rule. Fix $f, g \in \Gamma_0(X)$ and $x \in X$. Since $\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g$, it is not hard to see that $\operatorname{int}(\operatorname{dom}(f+g))$ coincides with $\operatorname{int}(\operatorname{dom} f) \cap \operatorname{int}(\operatorname{dom} g)$ and so

$$\beta\mathcal{P}_X(f+g,x) = \min\{\beta\mathcal{P}_X(f,x), \beta\mathcal{P}_X(f,x)\} \tag{2.13}$$

for each $x \notin \operatorname{int}(\operatorname{dom}(f+g))$. Thus, without loss of generality, we may assume that $\operatorname{int}(\operatorname{dom}(f+g)) \neq \emptyset$. Fix $x \in \operatorname{int}(\operatorname{dom}(f+g))$. Applying Proposition 1.2.22, we know that $\partial(f+g)(x) = \partial f(x) + \partial g(x)$, and so if one of the functions is not G-differentiable at $x$, then $\beta\mathcal{P}(f+g,x)=0$ and so equation 2.13 holds at $x$. Now, if both functions are G-differentiable at $x$ we have that for each $B\in\beta(X)$,

$$\begin{aligned}&\limsup_{t\searrow 0}\sup_{h\in B}\left|\frac{(f+g)(x+th)-(f+g)(x)}{t}-\nabla(f+g)(x)h\right|\\ \leq&\limsup_{t\searrow 0}\sup_{h\in B}\left|\frac{f(x+th)-f(x)}{t}-\nabla f(x)h\right|+\limsup_{t\searrow 0}\sup_{h\in B}\left|\frac{g(x+th)-g(x)}{t}-\nabla g(x)h\right|.\end{aligned}$$

Thus, if both functions are $\beta(X)$-differentiable at $x$, then $f+g$ also is $\beta(X)$-differentiable at $x$. Finally, assume that $f+g$ and one of the functions, let us say $f$, are $\beta(X)$-differentiable at $x$. To conclude that equation (2.13) always holds, it is enough to show that in this case $g$ must also be $\beta(X)$-differentiable at $x$.

Clearly $g$ has to be G-differentiable at $x$ since it was already established that this was a necessary condition in order to have the $\beta(X)$-differentiability of $f+g$ at $x$. Moreover, $\nabla g(x)=\nabla(f+g)(x)-\nabla f(x)$. Thus, denoting $\gamma:=f+g$, for each $B\in\beta(X)$ we can write

$$\begin{aligned}&\limsup_{t\searrow 0}\sup_{h\in B}\left|\frac{g(x+th)-g(x)}{t}-\nabla g(x)h\right|\\ \leq&\limsup_{t\searrow 0}\sup_{h\in B}\left|\frac{f(x+th)-f(x)}{t}-\nabla f(x)h\right|+\limsup_{t\searrow 0}\sup_{h\in B}\left|\frac{\gamma(x+th)-\gamma(x)}{t}-\nabla\gamma(x)h\right|,\end{aligned}$$

and so $g$ is also $\beta(X)$-differentiable at $x$. This finishes the proof. □

All those examples of convex smooth-like properties have an extra feature in common: *the families of structural and $w^*$-geometrical spaces are equivalent.* This is provided by proving that

$$X \text{ is a } \beta\text{-differentiability space} \Rightarrow X\times\mathbb{R} \text{ is a } \beta\text{-differentiability space.}$$

The proof is given for GD spaces, based on penalty functions (see [36, Proposition 6.5]), proposed by M. Fabian and using the ideas of A. Ioffe, as commented at the end of [36, Ch. 6].

We would like to introduce this implication as a characteristic of smooth-like properties, but we encounter a problem: Convex smooth-like and $w^*$-smooth-like properties are introduced to study SDPD Banach spaces (see Definition 1.3.17 in Chapter 1 and Chapter 3 below), but we still don't know if SDPD spaces are stable by product, even when one of them is finite-dimensional. These problems remain open.

# Chapter 3

# The Faces Radon-Nikodým Property

In this chapter, we come back to the study of the last part of Chapter 1, namely, the study of SDPD spaces. Noting that for a Banach space $X$ the Mackey topology $\tau(X^*, X^{**})$ in the dual space coincides with the topology of the dual norm and according to Lemma 2.1.1, the definition of SDPD spaces (Definition 1.3.17) can be redefined as follows: A Banach space $X$ is an SDPD space if for every function $f \in \Gamma_0(X^*, w^*)$ there exists a dense subset $D$ of $\operatorname{int}(\operatorname{dom} f)$ such that

$$\partial f(x^*) = \overline{X \cap \partial f(x^*)}^{w^{**}}, \tag{3.1}$$

for all $x^* \in D$. Our aim in this chapter is to study SDPD spaces according to the tools developed in Chapter 2, trying to answer the following fundamental question: Does the class of SDPD spaces coincides or not with the class of $w^*$-Asplund spaces?

## 3.1 Characterizations of SDPD equation

In this section, we will summarize the known characterizations of equation (3.1). Recall that in Proposition 1.3.18 we already showed that for every function $f \in \Gamma_0(X^*, w^*)$ and every $x^* \in \operatorname{int}(\operatorname{dom} f)$

$$\partial f(x^*) = \overline{X \cap \partial f(x^*)}^{w^{**}} \iff f'(x^*, \cdot) \text{ is } w^*\text{-lsc.}$$

Clearly, in order to have equation (3.1) for $f \in \Gamma_0(X^*, w^*)$ at $x^* \in \operatorname{int}(\operatorname{dom} f)$ it is necessary that $X \cap \partial f(x^*) \neq \emptyset$. A natural first question is if this is also sufficient. The answer is negative, as the following example shows:

**Example 3.1.1** Fix $X = \ell^1$ and denote by $e_i$ the $i$th canonic vector of $\ell^1$. Consider the two sequences $(x^n)$ and $(y^n)$ in $\ell^1$ defined as follows: $x^1 = y^1 = e_1$ and for $n > 0$

$$\begin{cases} x^n & := \frac{1}{n}e_1 + e_n. \\ \\ y^n & := \frac{1}{n}e_1 - e_n. \end{cases}$$

Let $K := \overline{\text{co}}\{x^n, y^n \ : \ n \in \mathbb{N}\}$. We will show that $0 \in \exp(K)$ (as a subset of $X$), but $0 \notin \text{ext}(\text{cl}_{w^{**}}(K))$ (as a subset of $X^{**}$). If our claim holds true, then there exists a functional $\varphi \in X^* = \ell^\infty$ such that $X \cap \partial\sigma_K(\varphi) = \{0\}$ but $\{0\} \subsetneq \partial\sigma_K(\varphi)$, since otherwise Lemma 1.3.14 would yield to the inclusion $\varphi \in \text{ext}(\text{cl}_{w^{**}}(K))$. After proving this, we would have shown that $\sigma_K$ and $\varphi$ don't verify equation (3.1), even though $\varphi$ is a support functional of $K$.

Recalling that $X^* = \ell^\infty$, let us fix $\varphi := -e_1$. For each $n \in \mathbb{N}$, we have that $\varphi(x^n) = \varphi(-y^n) = -1/n < 0$. Thus, $\varphi(x) \leq 0$ for each $x \in K$. Also, since $\frac{1}{n}e_1 = \frac{1}{2}x^n + \frac{1}{2}y^n \in K$, we know that $0 \in K$ and so $\sigma_K(\varphi) = 0 = \varphi(0)$.

Let us prove first that $\varphi$ exposes 0 in $K$. Suppose that this is not the case, that is, there exists $z \in K \setminus \{0\}$ such that $\varphi(z) = 0$. We know that there exists a sequence of convex combinations $(z^n)$ converging to $z$ given by

$$z^n = \sum_{i=1}^{k_n} \alpha_i^n x^i + \beta_i^n y^i, \qquad \text{with } \alpha_i^n, \beta_i^n \geq 0 \text{ and } \sum_{i=1}^{k_n} \alpha_i^n + \beta_i^n = 1.$$

Fix $j \in \mathbb{N}$, and consider the real sequence $(z_j^n)_{n\in\mathbb{N}}$ given by the $j$th coordinate of each $z^n$. Looking at the definitions of $x^n$, $y^n$ and $z^n$ we deduce that

$$z_j^n = \begin{cases} \alpha_j^n - \beta_j^n & \text{if } j \leq k_n \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality, let us assume that $k_n \geq j$ for each $n \in \mathbb{N}$. Fixing $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have the inequality

$$\sum_{i=1}^{k_n} \frac{\alpha_i + \beta_i}{i} = |\varphi(z^n)| \leq \frac{\varepsilon}{2j}.$$

This yields that $\alpha_i + \beta_i \leq \varepsilon/2$ and so, $\alpha_j \leq \varepsilon/2$ and $\beta_j \leq \varepsilon/2$. Then, for each $n \geq n_0$ we can write

$$|z_j^n| = |\alpha_j - \beta_j| \leq \varepsilon.$$

Since $\varepsilon$ is arbitrary, this entails that $z_j^n \to 0$ and so,

$$z_j = \langle e_j, z\rangle = \lim_{n\to\infty} \langle e_j, z^n\rangle = \lim_{n\to\infty} z_j^n = 0.$$

Repeating this process for each $j \in \mathbb{N}$, we deduce that $z = 0$, which is a contradiction. Thus, $\varphi$ exposes 0 in $K$.

To finish our example, we only need to prove that $0 \notin \mathrm{ext}(\mathrm{cl}_{w^{**}} K)$. Note first that $\frac{1}{n}e_1$ converges to 0 and, since $\mathbb{B}_{X^{**}}$ is $w^{**}$-compact, we also know that the sequence $(e_n)_{n\in\mathbb{N}}$ has a subnet $(u_\alpha)_{\alpha\in\Lambda}$ $w^{**}$-converging to some point $u^{**} \in \mathbb{B}_{X^{**}}$. If we consider the functional $\mathbb{1} \in \ell^\infty$ given by the constant sequence $\mathbb{1}_n = 1$, we easily see that $u^{**} \neq 0$, since

$$\langle \mathbb{1}, u^{**}\rangle = \lim_{\alpha} \langle \mathbb{1}, u_\alpha\rangle = 1.$$

Let us denote for each $\alpha \in \Lambda$ the integer $n_\alpha \in \mathbb{N}$ such that $u_\alpha = e_{n_\alpha}$. If we consider the nets $(x^\alpha)$ and $(y^\alpha)$ given by

$$x^\alpha = \frac{1}{n_\alpha}e_1 + u_\alpha \qquad \text{and} \qquad y^\alpha = \frac{1}{n_\alpha}e_1 - u_\alpha,$$

it is not hard to see that $x^\alpha \rightharpoonup^{w^{**}} u^{**}$ and $y^\alpha \rightharpoonup^{w^{**}} -u^{**}$, and so $u^{**}$ and $-u^{**}$ belong to $\mathrm{cl}_{w^{**}}(K)$. Thus, since $0 = \frac{1}{2}u^{**} + \frac{1}{2}(-u^{**})$, we deduce that 0 is not an extreme point of $\mathrm{cl}_{w^{**}}(K)$, which finishes the proof.

$\square$

In their paper of 2007 [13], Chakrabarty, Shunmunagaraj and Zălinescu studied several continuity properties of the subdifferential and $\varepsilon$-subdifferential of convex functions. Their work gather some other contributions in the same line, as [15], [27], [29] and [28], and it provides a survey as well as a generalization of those contributions.

In that paper, equation (3.1) was characterized in terms of a notion called *Hausdorff-upper semicontinuity.*

**Definition 3.1.2** *Let $(T, \tau)$ be a topological space, $(Z, \theta)$ be a locally convex space, $M : T \rightrightarrows Z$ be a set-valued operator between them and $t_0 \in T$. We say that $M$ is $\tau$-$\theta$* Hausdorff-upper semicontinuous *($\tau$-$\theta$* H-usc*, for short) at $t_0$ if*

$$\forall V \in \mathcal{N}_Z(0),\ \exists U \in \mathcal{N}_T(t_0) \text{ such that } M(U) \subseteq M(t_0) + V.$$

Note that, if $(Z, \theta)$ is a normed space, that is, there exists a norm $\|\cdot\|$ on $Z$ such that $\theta = \tau_{\|\cdot\|}$, then $M$ is $\tau$-$\tau_{\|\cdot\|}$ H-usc at $t_0$ if for every $\varepsilon > 0$ there exists $U \in \mathcal{N}_T(t_0)$ such that

$$M(U) \subseteq M(t_0) + B_Z(0, \varepsilon)(= \{z \in Z \ : \ d_{M(t_0)}(z) < \varepsilon\}).$$

Note also that Hausdorff-upper semicontinuity is weaker than usual upper-semicontinuity (see Definition 1.2.14).

**Definition 3.1.3** *For a function $f \in \Gamma_0(X^*, w^*)$, we define the set-valued operator $\eth f : \mathbb{R}_+ \times X^* \rightrightarrows X$ given by*

$$\eth f(\varepsilon, x^*) := X \cap \partial_\varepsilon f(x^*).$$

In [13], the set-valued operator $\eth f$ was denoted as $S_f^X$, but we prefer this new notation which gives the idea of a truncated $\varepsilon$-subdifferential. Note also that, endowing $X^*$ with the $w^*$-topology, $\eth f(\varepsilon, x^*)$ coincides with the $\varepsilon$-subdifferential of $f$ at $x^*$ with respect to the duality $\langle X, X^* \rangle$. In the next two propositions, we recall some results in [13] that we will need in the other sections.

**Proposition 3.1.4** ([13, Proposition 5.1]) *Let $\theta$ be a locally convex topology on $X$ in between $w(X, X^*)$ and $\tau_{\|\cdot\|}$, and let $f \in \Gamma_0(X^*, w^*)$ and $x^* \in \operatorname{int}(\operatorname{dom} f)$. The following assertions are equivalent:*

*(a)* $\eth f(\cdot, x^*) : \mathbb{R}_+ \rightrightarrows X$ *is* $\tau_0$-$\theta$ *H-usc at* $0$.

*(b)* $\eth f(0, \cdot) : X^* \rightrightarrows X$ *is* $\tau_{\|\cdot\|}$-$\theta$ *H-usc at* $x^*$.

*(c)* $\eth f$ *is* $(\tau_0 \times \tau_{\|\cdot\|})$-$\theta$ *H-usc at* $(0, x^*)$.

**Theorem 3.1.5** ([13, Proposition 5.2]) *Let $f \in \Gamma_0(X^*, w^*)$ and let $x^* \in \operatorname{int}(\operatorname{dom} f)$. We have that*

$$\partial f(x^*) = \overline{X \cap \partial f(x^*)}^{w^{**}} \iff \eth f(0, \cdot) \text{ is } \tau_{\|\cdot\|}\text{-}w \text{ H-usc at } x^*.$$

# 3.2 Geometrical interpretation of SDPD spaces: the FRNP

Let us consider the property $(\mathcal{P})$ as follows: For a Banach space $X$, the domain of $\mathcal{P}_X$ is $D(X) = \Gamma_0(X^*, w^*) \times X^*$ and $\mathcal{P}_X$ is given by the equivalence

$$\mathcal{P}_X(f, x^*) = 1 \iff x^* \in \operatorname{int}(\operatorname{dom} f^*) \text{ and } \partial f(x^*) = \overline{X \cap \partial f(x^*)}^{w^{**}}. \tag{3.2}$$

We will show that $(\mathcal{P})$ is a convex $w^*$-smooth-like property and thus, to be an SDPD space will be exactly to be a $(\mathcal{P})$-$w^*$-structural space. To do so, we will work with the characterization of Theorem 3.1.5 and we will prove that Hausdorff-upper semicontinuity satisfies the requirements of a convex $w^*$-smooth-like property.

In what follows, $\theta$ will be a locally convex topology on $X$ between $w(X, X^*)$ and $\tau_{\|\cdot\|}$.

**Lemma 3.2.1** *Let $f, g \in \Gamma_0(X^*, w^*)$ and $U \subseteq X^*$ be an open set. If $f$ and $g$ coincide in $U$, then for every $u^* \in U$*

$$\eth f(0, \cdot) \text{ is } \tau_{\|\cdot\|}\text{-}\theta \text{ H-usc at } u^* \iff \eth g(0, \cdot) \text{ is } \tau_{\|\cdot\|}\text{-}\theta \text{ H-usc at } u^*.$$

*Proof.* Note that, since $f\big|_U = g\big|_U$, we also have that for each $u^* \in U$, $\partial f(u^*) = \partial g(u^*)$ and therefore $\eth f(0, u^*) = \eth g(0, u^*)$. Choose $u^* \in U$ and suppose that $\eth f(0, \cdot)$ is $\tau_{\|\cdot\|}$-$\theta$

H-usc at $u^*$. Let $V \in \mathcal{N}(0,\theta)$. There exists $U' \in \mathcal{N}(u^*, \tau_{\|\cdot\|_*})$ such that

$$\eth f(0,U') \subseteq \eth f(0,u^*) + V.$$

Define $U'' = U' \cap U$, which is also a neighborhood of $u^*$. Then,

$$\eth g(0,U'') = \eth f(0,U'') \subseteq \eth f(0,u^*) + V = \eth g(0,u^*) + V.$$

Since $V$ is an arbitrary element of $\mathcal{N}_X(0,\theta)$, we conclude that $g$ has to be $\tau_{\|\cdot\|}$-$\theta$ H-usc at $u^*$. The equivalence follows from the symmetry of $f$ and $g$ in the previous reasoning. $\square$

**Lemma 3.2.2** *Let $\sigma \in \Gamma_0(X^*, w^*)$ be a sublinear function, $x^* \in \operatorname{int}(\operatorname{dom}\sigma)$. Then*

$$\eth\sigma(0,\cdot) \text{ is } \tau_{\|\cdot\|}\text{-}\theta \text{ H-usc at } x^* \implies \eth\sigma(0,\cdot) \text{ is } \tau_{\|\cdot\|}\text{-}\theta \text{ H-usc at } tx^*,\ \forall t>0.$$

*Also, if $\sigma = \sigma_K$ with $K \subseteq X$ being a closed convex set, then for each $x \in X$*

$$\eth\sigma_K(0,\cdot) \text{ is } \tau_{\|\cdot\|}\text{-}\theta \text{ H-usc at } x^* \implies \eth\sigma_{K+x}(0,\cdot) \text{ is } \tau_{\|\cdot\|}\text{-}\theta \text{ H-usc at } x^*.$$

*Proof.* For the first implication, let $t > 0$ and suppose that there exists an open neighborhood $V \in \mathcal{N}(0,\theta)$ such that

$$\forall U \in \mathcal{N}(tx^*),\ \eth\sigma(0,U) \not\subseteq \eth\sigma(0,tx^*) + V.$$

Then, we can construct a net $(x_U^*)_{U\in\mathcal{N}(tx^*)}$ such that $x_U^* \to tx^*$ and

$$\forall U \in \mathcal{N}(tx^*),\ \exists x_U \in X,\ x_U \in \eth\sigma(0,x_U^*) \setminus (\eth\sigma(0,tx^*) + V).$$

Recalling that $\sigma$ is sublinear, we get that for each $z^* \in X^*$, $\partial\sigma(z^*) = \partial\sigma(tz^*)$ and therefore $\eth\sigma(0,z^*) = \eth(0,tz^*)$. Then, since the net $(t^{-1}x_U^*)_{U\in\mathcal{N}(tx^*)}$ converges to $x^*$ and $\eth\sigma(0,\cdot)$ is $\tau_{\|\cdot\|}$-$\theta$ H-usc at $x^*$, there exists $U_0 \in \mathcal{N}(tx^*)$ such that

$$\forall U \in \mathcal{N}(tx^*) \text{ with } U \subseteq U_0,\ \eth\sigma(0,t^{-1}x_U^*) \subseteq \eth\sigma(0,x^*) + V = \eth\sigma(0,tx^*) + V.$$

In particular, $x_{U_0} \in \eth\sigma(0,x_{U_0}^*) = \eth(0,t^{-1}x_{U_0}^*) \subseteq \eth\sigma(0,tx^*) + V$, which is a contradiction.

To prove the second implication, note that for each $x \in X$ and each $\varepsilon \geq 0$, $\partial_\varepsilon\sigma_{K+x}(z^*) = \partial_\varepsilon\sigma_K(z^*) + x$ for all $z^* \in X^*$. Therefore,

$$\eth\sigma_{K+x}(\varepsilon,z^*) = \eth\sigma_K(\varepsilon,z^*) + x,\ \forall(\varepsilon,z^*) \in \mathbb{R}_+ \times X^*.$$

Now, if $\eth\sigma_K(0,\cdot)$ is $\tau_{\|\cdot\|}$-$\theta$ H-usc at $x^*$, then, by Proposition 3.1.4, we get that $\eth\sigma_K(\cdot,x^*)$ is $\tau_0$-$\theta$ H-usc at 0. Let $V \in \mathcal{N}(0,\theta)$. There exists $\varepsilon > 0$ such that

$$\eth\sigma_K(\varepsilon,x^*) \subseteq \eth\sigma_K(0,x^*) + V.$$

Thus,

$$\eth\sigma_{K+x}(\varepsilon,x^*) = \eth\sigma_K(\varepsilon,x^*) + x \subseteq \eth\sigma_K(0,x^*) + x + V = \eth\sigma_{K+x}(0,x^*) + V.$$

We conclude that $\sigma_{K+x}(\cdot,x^*)$ is $\tau_0$-$\theta$ H-usc at 0 and, applying again Proposition 3.1.4, the proof is completed. $\square$

**Lemma 3.2.3** *Let* $f \in \Gamma_0(X)$ *with* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$ *and* $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. *Then*

$$\eth f^*(0,\cdot) \text{ is } \tau_{\|\cdot\|}\text{-}\theta \text{ H-usc at } x^* \iff \eth\sigma_{\operatorname{epi} f}(0,\cdot) \text{ is } (\tau_{\|\cdot\|}\times\tau_0)\text{-}(\theta\times\tau_0) \text{ H-usc at } (x^*,-1).$$

*Proof.* Let us show first the necessity. Suppose that $\eth\sigma_{\operatorname{epi} f}(0,\cdot)$ is $(\tau_{\|\cdot\|}\times\tau_0)$-$(\theta\times\tau_0)$ H-usc at $(x^*,-1)$. Then, for each $V \in \mathcal{N}_X(0,\theta)$ there exists $\varepsilon > 0$ such that for $U = (x^*,-1)+\varepsilon(\mathbb{B}_{X^*}\times[-1,1])$ we have that

$$\eth\sigma_{\operatorname{epi} f}(0,U) \subseteq \eth\sigma_{\operatorname{epi} f}(0,(x^*,-1)) + V\times\mathbb{R}.$$

In particular, for each $z^* \in x^*+\varepsilon\mathbb{B}_{X^*}$, $\eth\sigma_{\operatorname{epi} f}(0,(z^*,-1)) \subseteq \eth\sigma_{\operatorname{epi} f}(0,(x^*,-1)) + V\times\mathbb{R}$. Noting that

$$\eth\sigma_{\operatorname{epi} f}(0,(z^*,-1)) = \{(z,f(z)) \in \operatorname{epi} f \ : \ z \in \eth f^*(0,z^*)\}, \tag{3.3}$$

we conclude that $\eth f^*(0,z^*) \subseteq \eth f^*(0,x^*)+V$ for each $z^* \in x^*+\varepsilon\mathbb{B}_{X^*}$. Thus, $\eth f^*(0,\cdot)$ is $\tau_{\|\cdot\|}$-$\theta$ H-usc at $x^*$.

For the sufficiency, suppose that $\eth f^*(0,\cdot)$ is $\tau_{\|\cdot\|}$-$\theta$ H-usc at $x^*$. Let $V \in \mathcal{N}(0,\theta)$ and $\delta > 0$. Since $\theta$ is finer than the weak topology, we can assume without losing generality that

$$V \subseteq \left\{x \in X \ : \ |\langle x^*,x\rangle| \leq \tfrac{\delta}{2}\right\}.$$

By hypothesis, there exists $\varepsilon_1 > 0$ such that $\eth f^*(0,x^*+\varepsilon_2\mathbb{B}_{X^*}) \subseteq \eth f^*(0,x^*)+V$. Also, since $\partial f^*$ is bounded near $x^*$ we can apply Lemma 2.3.5 to find $\varepsilon_2 > 0$ such that

$$\forall z^* \in x^*+\varepsilon_2\mathbb{B}_{X^*},\ \eth f^*(0,z^*) \subseteq \partial_{\delta/2} f^*(x^*).$$

Define $\varepsilon = \min\{\varepsilon_1,\varepsilon_2,\delta\}$. Then, for $z^* \in x^*+\varepsilon\mathbb{B}_{X^*}$ and $z \in \eth f^*(0,z^*)$, there exists $x \in \eth f^*(0,x^*)$ such that $z-x \in V$, and so, since $x^* \in \partial f(x)$ and $x^* \in \partial_{\delta/2} f(z)$ (since $f \in \Gamma_0(X)$), we have that

$$-\delta < -\frac{\delta}{2} \leq \langle x^*, z-x\rangle \leq f(z)-f(x) \leq \langle x^*,z-x\rangle + \frac{\delta}{2} \leq \delta.$$

Thus, we have that $(z,f(z)) \in (x,f(x))+V\times\,]-\delta,\delta[\,\subseteq \eth\sigma_{\operatorname{epi} f}(0,(x^*,-1))+V\times\,]-\delta,\delta[$, where the last inclusion is due to the fact that $x \in \eth f^*(0,x^*)$ and the equality (3.3), which is valid for all $z^* \in X^*$. Using again equality (3.3), we can write

$$\forall z^* \in x^*+\varepsilon\mathbb{B}_{X^*},\ \eth\sigma_{\operatorname{epi} f}(0,(z^*,-1)) \subseteq \eth\sigma_{\operatorname{epi} f}(0,(x^*,-1))+V\times\,]-\delta,\delta[.$$

Now, take $\eta \in\, ]0,1[$ small enough such that

$$\left]\frac{1}{1+\eta},\frac{1}{1-\eta}\right[\cdot(x^*+\eta\mathbb{B}_{X^*}) \subseteq x^*+\varepsilon\mathbb{B}_{X^*},$$

and define $U = (x^*+\eta\mathbb{B}_{X^*})\times\,]-1-\eta,-1+\eta[$. For $(z^*,t) \in U$, we have that $\frac{1}{|t|} \in \left]\frac{1}{1+\eta},\frac{1}{1-\eta}\right[$, and so $|t|^{-1}z^* \in x^*+\varepsilon\mathbb{B}_{X^*}$. Then,

$$\eth\sigma_{\operatorname{epi} f}(0,(z^*,t)) = \eth\sigma_{\operatorname{epi} f}(0,(|t|^{-1}z^*,-1)) \subseteq \eth\sigma_{\operatorname{epi} f}(0,(x^*,-1))+V\times\,]-\delta,\delta[,$$

and so, we have that $\eth\sigma_{\operatorname{epi} f}(0,U) \subseteq \eth\sigma_{\operatorname{epi} f}(0,(x^*,-1)) + V\times ]-\delta,\delta[$. Since the sets of the form $V\times]-\delta,\delta[$ are a base of neighborhoods of 0 for the topology $\tau\times\tau_0$, we conclude that $\eth\sigma_{\operatorname{epi} f}(0,\cdot)$ is $(\tau_{\|\cdot\|}\times\tau_0)$-$(\theta\times\tau_0)$ H-usc at $(x^*,-1)$. □

**Lemma 3.2.4** *Let $Y$ be a Banach space such that there exists $T: Y\to X$ an one-to-one bounded linear operator with closed range. Let $\theta_Y$ be a locally convex topology on $Y$ between the weak and the norm topologies. If $T$ is $\theta_Y$-$\theta$-continuous and $\theta_Y$-$\left(\theta\big|_{T(Y)}\right)$-open, then, for every function $f\in\Gamma_0(Y^*,w^*)$ and every functional $y^*\in Y^*$,*

$$\eth(f\circ T^*)(0,\cdot) \text{ is } \tau_{\|\cdot\|}\text{-}\theta \text{ H-usc at } x^* \iff \eth f(0,\cdot) \text{ is } \tau_{\|\cdot\|}\text{-}\theta_Y \text{ H-usc at } T^*y^*.$$

*Proof.* Fix $f\in\Gamma_0(Y^*,w^*)$ and $y^*\in Y^*$. Note that, by the classical chain rule of the subdifferential, for each $z^*\in X$

$$\partial(f\circ T^*)(z^*) = T^{**}[\partial f(T^*z^*)],$$

and therefore $\eth(f\circ T^*)(0,z^*) = T[\eth f(0,T^*z^*)]$, according to the fact that for each set $A\subset Y^{**}$, we have that $T(Y\cap A) = X\cap T^{**}(A)$. Indeed, since $T^{**}\big|_Y = T$, we have that $T(Y\cap A)\subseteq X\cap T^{**}(A)$. For the other inclusion, let $x\in X\cap T^{**}(A)$ and $y^{**}\in A$ such that $T^{**}(y^{**}) = x$. Since the range of $T^{**}$ is $\overline{T(Y)}^{w^{**}}$, we have that $x\in\overline{T(Y)}^{w^{**}}$, and so there exists a net $(x_\lambda)_{\lambda\in\Lambda}\subseteq T(Y)$ such that $x_\lambda \rightharpoonup x$. But, since $T(Y)$ is convex and closed, it is also $w$-closed. Then, $x\in T(Y)$ and so, by injectivity of $T^{**}$, $y^{**}\in Y$. Therefore, $x\in T(Y\cap A)$.

This latter equality will be the key tool for the proof.

Assume first that $\eth f(0,\cdot)$ is $\tau_{\|\cdot\|}$-$\theta_Y$ H-usc at $T^*x^*$ and let $W\in\mathcal{N}_X(0,\theta)$. Since $T^{-1}(W)\in\mathcal{N}_Y(0,\theta_Y)$, there exists $U\subseteq\mathcal{N}_{Y^*}(T^*x^*)$ such that

$$\eth f(0,U)\subseteq \eth f(0,T^*x^*) + T^{-1}(W).$$

Therefore,

$$\eth(f\circ T^*)(0,(T^*)^{-1}(U)) = T[\eth f(0,U)] \subseteq T[\eth f(0,T^*x^*)] + T[T^{-1}(W)] \subseteq \eth(f\circ T^*)(0,x^*) + W,$$

where the first equality is provided by the fact that $T^*$ is onto (and so $T^*[(T^*)^{-1}(U)] = U$). By the arbitrariness of $W$ and the fact that $T^*$ is an open map, we conclude that $\partial(f\circ T^*)(0,\cdot)$ is $\tau_{\|\cdot\|}$-$\theta$ H-usc at $x^*$.

Assume now that $\eth(f\circ T^*)(0,\cdot)$ is $\tau_{\|\cdot\|}$-$\theta$ H-usc at $x^*$ and let $V\in\mathcal{N}_Y(0,\theta_Y)$. We can assume, without losing generality, that $V$ is open and so, that $T(V)$ is a $\theta\big|_{T(Y)}$-open set of $T(Y)$. Then, there exists an open set $W\in\mathcal{N}_X(0,\theta)$ such that $W\cap T(Y) = T(V)$ and so, there also exists $U\subseteq\mathcal{N}_{X^*}(x^*)$ such that

$$\eth(f\circ T^*)(0,U)\subseteq \eth(f\circ T^*)(0,x^*) + W.$$

Thus, we have that $T[\eth f(0, T^*(U))] \subseteq T[\eth f(0, T^*x^*)] + W$. But, since $T[\eth f(0, T^*(U))] \subseteq T(Y)$,

$$T[\eth f(0, T^*(U))] \subseteq T(Y) \cap (T[\eth f(0, T^*x^*)] + W) = T[\eth f(0, T^*x^*) + V].$$

Finally, since $T$ is one-to-one, $\eth f(0, T^*(U)) \subseteq \eth f(0, T^*x^*) + V$, and also, since $T^*$ is onto, $T^*(U) \in \mathcal{N}_{Y^*}(T^*x^*)$. Since $V$ is an arbitrary element of $\mathcal{N}_Y(0, \theta_Y)$, we conclude that $\eth f(0, \cdot)$ is $\tau_{\|\cdot\|}$-$\theta$ H-usc at $T^*x^*$. This finishes the proof. □

We can now establish that $(\mathcal{P})$ is a convex $w^*$-smooth-like property.

**Theorem 3.2.5** *The property $(\mathcal{P})$ is a convex $w^*$-smooth-like property, and so, in view of the equivalence (3.2), to be an SDPD space is to be $(\mathcal{P})$-$w^*$-structural.*

*Proof.* If we consider $\theta = w$, Theorem 3.1.5 and Lemmas 3.2.1, 3.2.2 and 3.2.3 show that $(\mathcal{P})$ is local, set-consistent and epigraphical, respectively. Thus, it only rests to prove that $(\mathcal{P})$ is $w^*$-transitive.

For this, let $X, Y$ be two Banach spaces such that there exists $T : Y \to X$ an one-to-one bounded linear operator. By Lemma 3.2.4, we only need to prove that $T$ is $w$-$w\big|_{T(Y)}$-open, since it is known that it is $w$-$w$-continuous. Let then $W$ be a $w$-open set of $Y$. Without losing generality, we may assume that $0 \in W$ and

$$W = \{y \in Y \ : \ |\langle y_i^*, y\rangle| \leq \varepsilon_i, \ \forall i \in \{1, \dots, n\}\},$$

where $n \in \mathbb{N}$, $y_1^*, \dots, y_n^* \in Y^*$ and $\varepsilon_1, \dots, \varepsilon_n > 0$. Since $T$ is one-to-one and has closed range, $T^*$ is onto and so there exist $x_1^*, \dots, x_n^* \in X^*$ such that $T^*(x_i^*) = y_i^*$, for all $i \in \{1, \dots, n\}$. Then, we can write

$$\begin{aligned} W &= \{y \in Y \ : \ |\langle T^*x_i^*, y\rangle| \leq \varepsilon_i, \ \forall i \in \{1, \dots, n\}\} \\ &= \{y \in Y \ : \ |\langle x_i^*, Ty\rangle| \leq \varepsilon_i, \ \forall i \in \{1, \dots, n\}\}. \end{aligned}$$

Let $V = \{x \in X \ : \ |\langle x_i^*, x\rangle| \leq \varepsilon_i, \ \forall i \in \{1, \dots, n\}\}$. It is clear that $W = T^{-1}(V) = T^{-1}(V \cap T(Y))$, and since $T$ is a bijection between $Y$ and $T(Y)$, we conclude that $T(W) = V \cap T(Y) \in w\big|_{T(Y)}$, finishing the proof. □

Now we will give a characterization of SDPD spaces via "exposed faces". Even though there are several definitions in the literature of what a face of a set is, we will adopt the convention that, in a Banach space $X$, a face $F$ of a set $K \subseteq X$ is an *exposed subset*, namely, there exists a functional $x^* \in X^*$ such that

$$F = \{x \in K \ : \ \langle x^*, x\rangle = \sigma_K(x^*)\}.$$

In such a case, we will write $F = F[K, x^*]$ to avoid ambiguity whenever is needed.

**Definition 3.2.6** ($\theta$-exposed Faces) *Let $K \subseteq X$ be a closed convex set, $F$ be a face of $K$ and $\theta$ be locally convex topology on $X$ between the $w(X, X^*)$ and $\tau_{\|\cdot\|}$ topologies. We will say that $F$ is* $\theta$-exposed *by $x^* \in X^*$ if $F = F[K, x^*]$ and for each $\theta$-neighborhood $V$ of $0$, there exists $\alpha > 0$ such that*

$$S(K, x^*, \alpha) \subset F + V.$$

*The set of* $\theta$-exposing functionals *$x^* \in X^*$, namely those which $\theta$-expose a face of $K$, will be denoted by $E[K, \theta]$.*

**Definition 3.2.7** (Faces Radon-Nikodým Property) *A Banach space $X$ is said to have the **Faces Radon-Nikodým property** (FRNP, for short) if for each closed convex bounded set $K$ of $X$, $E[K, w]$ is dense in $X^*$.*

**Proposition 3.2.8** *For a convex closed set $K$ of $X$, $\mathcal{P}_X[K] = E[K, w]$, and so, to have the FRNP is to be $(\mathcal{P})$-geometrical.*

*Proof.* Let $K$ be a closed convex set of $X$. Note that for any $x^* \in \operatorname{dom} \sigma_K$ and any $\alpha > 0$, we have that $F[K, x^*] = \eth\sigma_K(0, x^*)$ and also

$$S(K, x^*, \alpha) = \bigcup_{\delta < \alpha} \eth\sigma_K(\delta, x^*).$$

Therefore, for any $V \in \mathcal{N}(0, w)$ we have that

$$\begin{aligned} S(K, x^*, \alpha) \subseteq F[K, x^*] + V &\iff \bigcup_{\delta<\alpha} \eth\sigma_K(\delta, x^*) \subseteq \eth(0, x^*) + V \\ &\iff \eth\sigma_K((0, \alpha), x^*) \subseteq \eth\sigma_K(0, x^*) + V. \end{aligned}$$

Thus, $x^* \in E[K, w]$ if and only if $\eth\sigma_K(\cdot, x^*)$ is $\tau_0$-$w$ H-usc at 0. The conclusion follows from Proposition 3.1.4.

□

**Example 3.2.9** The unit ball of $L^1[0, 1]$ with the usual norm satisfies $E[\mathbb{B}_{L^1[0,1]}, w]$ is dense in $X^*$.

*Proof.* It is known (see [50]) that $(L^1[0, 1])^* = L^\infty[0, 1]$ and that $(L^1[0, 1])^{**} = \mathcal{M}[0, 1]$, where $\mathcal{M}[0, 1]$ stands for the space of all bounded finite measures over the Borel $\sigma$-algebra, $\mathcal{B}[0, 1]$, which vanish at each $\lambda$-null set (where $\lambda$ is the Lebesgue measure in $[0, 1]$). Even more, the bidual norm of $\|\cdot\|_1$ (which we will denote simply by $\|\cdot\|$) over $\mathcal{M}[0, 1]$ is given by

$$\|m\| = |m|([0, 1]),$$

where $|m|$ stands for the total variation of $m$. Therefore, for $E \in \mathcal{B}[0, 1]$ with $\lambda(E) > 0$, we have that

$$\begin{aligned} \partial\|\cdot\|_\infty(\mathbb{1}_E) &= \{m \in \mathbb{B}_{\mathcal{M}[0,1]} \ : \ \langle m, \mathbb{1}_E\rangle = \|\mathbb{1}_E\|_\infty\} \\ &= \{m \in \mathcal{M}[0, 1] \ : \ m(E) = |m|([0, 1]) = 1\}, \end{aligned}$$

where $\mathbb{1}_E$ is the indicator function of $E$, in the sense of measure theory: Namely,

$$\forall t \in [0,1],\ \mathbb{1}_E(t) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{if } t \notin E. \end{cases}$$

We will prove that $\xi([0,1]) \subseteq E\left[\mathbb{B}_{L^\infty[0,1]}, w\right]$, where $\xi([0,1])$ is the set of simple functions over $[0,1]$, which is known to be dense in $L^\infty[0,1]$. Let first choose $E \in \mathcal{B}[0,1]$ with $\lambda(E) > 0$ and fix $m \in \partial\|\cdot\|_\infty(\mathbb{1}_E)$. It is known (see [50, Theorem 1.12]) that $m = m^+ - m^-$ and $|m| = m^+ + m^-$, where $m^+$ and $m^-$ are the positive and negative parts of $m$, respectively. Then, we can write

$$\begin{aligned} 1 = |m|([0,1]) &= m^+([0,1]) + m^-([0,1]) \\ &\geq m^+(E) + m^-([0,1]) \\ &\geq m(E) + m^-([0,1]) = 1 + m^-([0,1]). \end{aligned}$$

Therefore, $m^- \equiv 0$ and so, $m$ is positive. Also, we get that $m$ vanishes on $[0,1] \setminus E$. In particular, identifying $L^1(E)$ and $\mathcal{M}(E)$ as subspaces of $L^1[0,1]$ and $\mathcal{M}[0,1]$, we have that $m \in \mathcal{M}(E) = (L^1(E))^{**}$. Now, by Goldstine's theorem, there exists a net $(f_\alpha)_{\alpha\in A}$ in $L^1(E)$ with $\|f_\alpha\|_1 \leq 1$ for each $\alpha \in A$, such that

$$f_\alpha \rightharpoonup^{w^{**}} m.$$

Let us consider separately the nets $(f_\alpha^+)$ and $(f_\alpha^-)$. Since $\|f_\alpha^+\|_1, \|f_\alpha^-\|_1 \leq \|f_\alpha\|$ we have, by Alaoglu's theorem, that there exist $m_1, m_2 \in \mathbb{B}_{\mathcal{M}(E)}$ and two subnets $(f_\gamma^+)_{\gamma\in\Gamma}$, $(f_\gamma^-)_{\gamma\in\Gamma}$ of $(f_\alpha^+)$ and $(f_\alpha^-)$ respectively (which we can index by the same directed set $\Gamma$ just taking them sequentially), such that

$$f_\gamma^+ \rightharpoonup^{w^{**}} m_1 \quad \text{and} \quad f_\gamma^- \rightharpoonup^{w^{**}} m_2.$$

Since $f_\gamma \rightharpoonup^{w^{**}} m$, it is clear that $m = m_1 - m_2 \leq m_1$, and so $m_1(E) = 1$. Then, $m_2(E) = 0$ and since $m_2$ is positive, $m_2 \equiv 0$. We conclude then that $m = m_1$ and therefore $f_\gamma^+ \rightharpoonup^{w^{**}} m$. Let us consider then the net $(g_\gamma)_{\gamma\in A}$ given by

$$g_\gamma = \frac{1}{\|f_\gamma^+\|_1} f_\gamma^+.$$

Since $\|f_\gamma^+\|_1 = \int_E f_\gamma^+ d\lambda \to m(E) = 1$, we have that $g_\gamma \rightharpoonup^{w^{**}} m$. It is directly verifiable that $\|g_\gamma\|_1 = \int_E g_\gamma = 1$ for each $\gamma \in A$, and therefore, $(g_\gamma)_{\gamma\in A} \subseteq L^1(E) \cap \partial\|\cdot\|_\infty(\mathbb{1}_E) \subseteq L^1[0,1] \cap \partial\|\cdot\|_\infty(\mathbb{1}_E)$. We conclude that

$$\partial\|\cdot\|_\infty(\mathbb{1}_E) = \overline{L^1[0,1] \cap \partial\|\cdot\|_\infty(\mathbb{1}_E)}^{w^{**}},$$

and so $\mathbb{1}_E \in E\left[\mathbb{B}_{L^1[0,1]}, w\right]$, according to Proposition 3.2.8. Symmetrically, we can prove that $-\mathbb{1}_E \in E\left[\mathbb{B}_{L^1[0,1]}, w\right]$.

Choose now $f \in \xi([0,1])$ with $\|f\|_\infty = 1$ and fix $m \in \partial\|\cdot\|_\infty(f)$. Note that

$$|m|([0,1]) = \int f dm = \int f dm^+ - \int f dm^- \leq \int f^+ dm^+ + \int f^- dm^- \leq |m|([0,1]),$$

and therefore $m^+[f<0]=m^-[f>0]=0$. Defining the sets

$$E^+=\{t\ :\ f(t)=1\}\quad \text{and}\quad E^-=\{t\ :\ f(t)=-1\},$$

we claim that $m^+$ vanishes in $[0,1]\setminus E^+$ and $m^-$ vanishes in $[0,1]\setminus E^-$. Indeed, if the claim doesn't hold, we can assume without losing generality that there exists $F\subseteq[0,1]\setminus E^+$ such that $m^+(F)>0$. Let $\delta=\|f^+|_F\|_\infty<1$ (provided that $f$ is a simple function). Then

$$\int f^+dm^+=\int_F f^+dm^+ +\int_{[0,1]\setminus F} f^+dm^+\leq \delta m^+(F)+m^+([0,1]\setminus F)<m^+([0,1]),$$

and since $\int f^-dm^-\leq m^-([0,1])$, we would conclude that

$$\int fdm\leq\int f^+dm^+ +\int f^-dm^- < m^+([0,1])+m^-([0,1])=|m|([0,1]),$$

which is a contradiction. Therefore the claim holds and so $m^+\in\mathcal{M}(E^+)$ and $m^-\in\mathcal{M}(E^-)$. As we already proved, we can construct two nets $(g_\alpha^+)_{\alpha\in A}$ and $(g_\alpha^-)_{\alpha\in A}$ (that, up subnets, we can assume to be indexed by the same directed set) such that

1. $g_\alpha^+$ is positive, it vanishes in $[0,1]\setminus E^+$ and
$$\int_{E^+} g_\alpha^+d\lambda=\|g_\alpha^+\|_1=m^+(E^+)=m^+([0,1]).$$

2. $g_\alpha^-$ is positive, it vanishes in $[0,1]\setminus E^-$ and
$$\int_{E^-} g_\alpha^-d\lambda=\|g_\alpha^-\|_1=m^-(E^-)=m^-([0,1]).$$

3. $g_\alpha^+\rightharpoonup^{w^{**}} m^+$ and $g_\alpha^-\rightharpoonup^{w^{**}} m^-$.

Then, defining $g_\alpha=g_\alpha^+-g_\alpha^-$ for each $\alpha\in A$, we conclude that $g_\alpha\rightharpoonup^{w^{**}} m$ and

$$\|g_\alpha\|_1=\int_{E^+} g_\alpha^+d\lambda+\int_{E^+} g_\alpha^-d\lambda=m^+([0,1])+m^-([0,1])=|m|([0,1])=1,\ \forall\alpha\in A.$$

Finally, noting that

$$\int fg_\alpha d\lambda=\int_{E^+} fg_\alpha^+d\lambda-\int_{E^+} fg_\alpha^-d\lambda=\int_{E^+} g_\alpha^+d\lambda+\int_{E^+} g_\alpha^-d\lambda=1=\|f\|_\infty,$$

we have that $(g_\alpha)\subseteq L^1[0,1]\cap\partial\|\cdot\|_\infty(f)$, and thus, since $m$ is an arbitrary element of $\partial\|\cdot\|_\infty(f)$,

$$\partial\|\cdot\|_\infty(f)=\overline{L^1[0,1]\cap\partial\|\cdot\|_\infty(f)}^{w^{**}},$$

which finishes the proof, according to Proposition 3.2.8. □

**Proposition 3.2.10** *The Banach space $X$ has the FRNP if and only if for every $K \subseteq X$ nonempty convex bounded closed set, every $x^* \in X^*$ and every $\alpha > 0$ there exists $z^* \in E[K, w]$ such that*

$$F[K, z^*] \subseteq S(K, x^*, \alpha).$$

*Proof.*

$\Rightarrow$) Let $K$ be a nonempty convex bounded closed set of $X$, $x^* \in X^*$ and $\alpha > 0$. By Lemma 2.3.5, there exists $\varepsilon > 0$, such that for each $y^* \in x^* + \varepsilon\mathbb{B}_{X^*}$,

$$S(K, y^*, \tfrac{\alpha}{2}) \subseteq S(K, x^*, \alpha).$$

Since $X$ has the FRNP, there exists $z^* \in E[K, w] \cap (x^* + \varepsilon\mathbb{B}_{X^*})$. Thus,

$$F[K, z^*] = X \cap \partial\sigma_K(z^*) \subseteq S(K, z^*, \tfrac{\alpha}{2}) \subseteq S(K, x^*, \alpha),$$

which proves the implication.

$\Leftarrow$) Let $K_1$ be a convex closed set with $\operatorname{int}(\operatorname{dom}\sigma_{K_1}) \neq \emptyset$. We want to show, according to Theorem 2.3.20, that $E[K_1, w] \cap \operatorname{int}(\operatorname{dom}\sigma_{K_1})$ is nonempty. To do so, choose $x^* \in \operatorname{int}(\operatorname{dom}\sigma_{K_1})$ and define $K_2 = \eth\sigma_{K_1}(1, x^*)$. Since $x^* \in \operatorname{int}(\operatorname{dom}\sigma_{K_1})$, $K_2$ is bounded, and so, there exists $z^* \in E[K_2, w]$ such that

$$F[K_2, z^*] \subseteq S(K_2, x^*, \tfrac{1}{2}) = S(K_1, x^*, \tfrac{1}{2}),$$

where the last equality comes from the observation that

$$S(K_2, x^*, \tfrac{1}{2}) = K_2 \cap S(K_1, x^*, \tfrac{1}{2}),$$

according to the inclusion $K_2 \subseteq K_1$. To simplify notation, let $\mathbf{K}_i = \overline{K_i}^{w^{**}}$ (for $i = 1, 2$). We claim that $\partial\sigma_{K_2}(z^*) = \partial\sigma_{K_1}(z^*)$. To prove this, note first that for all $z_1^{**} \in \mathbf{K}_1 \setminus \mathbf{K}_2$, $\langle z_1^{**}, z^*\rangle < \sigma_{K_2}(z^*)$. If not, choose any $z_2^{**} \in \sigma_{K_2}(z^*)$. Since $\langle z_1^{**}, x^*\rangle < \sigma_{K_1}(x^*) - 1$ and $\langle z_2^{**}, x^*\rangle \geq \sigma_{K_1}(x^*) - \frac{1}{2}$, there exists $t \in (0, 1)$ such that $\langle z^{**}, x^*\rangle = \sigma_{K_1}(x^*) - \frac{3}{4}$, where $z^{**} = tz_1^{**} + (1-t)z_2^{**}$. Then, $z^{**} \in \mathbf{K}_2$ and

$$\langle z^{**}, z^*\rangle = t\langle z_1^{**}, x^*\rangle + (1-t)\langle z_2^{**}, x^*\rangle = t\sigma_{K_1}(z^*) + (1-t)\sigma_{K_2}(z^*) \geq \sigma_{K_2}(z^*).$$

This implies that $z^{**} \in \partial\sigma_{K_2}(z^*)$, which is a contradiction since

$$\partial\sigma_{K_2}(z^*) = \overline{F[K_2, z^*]}^{w^{**}} \subseteq \partial_{1/2}\sigma_{K_1}(x^*).$$

Now, suppose that there exists $z^{**} \in \partial\sigma_{K_1}(z^*) \setminus \partial_{K_2}(z^*)$. Then

$$\langle z^{**}, z^*\rangle = \sigma_{K_1}(z^*) \geq \sigma_{K_2}(z^*),$$

and so, since $z^{**} \notin \partial\sigma_{K_2}(z^*)$, we have that $z^{**} \in \mathbf{K}_1 \setminus \mathbf{K}_2$. But in this case we have shown that $\langle z^{**}, z^*\rangle < \sigma_{K_2}(z^*)$, which is a contradiction.

Therefore, $\partial\sigma_{K_1}(z^*) \subseteq \partial\sigma_{K_2}(z^*)$. Now, there are two possibilities: $\partial\sigma_{K_1}(z^*) = \partial\sigma_{K_2}(z^*)$ as we claimed, or $\partial\sigma_{K_1}(z^*) = \emptyset$. But $\partial\sigma_{K_1}(z^*)$ can not be empty, since if it were so, $z^*$ would be unbounded in $\mathbf{K}_1$ and there would exist $z^{**} \in \mathbf{K}_1 \setminus \mathbf{K}_2$ with $\langle z^{**}, z^*\rangle > \sigma_{K_2}(z^*)$ which is, as we know, a contradiction. Then, $\partial\sigma_{K_1}(z^*) = \partial\sigma_{K_2}(z^*)$, as we claimed. Even more, since $\partial\sigma_{K_1}(z^*)$ is nonempty and bounded, $z^* \in \operatorname{int}(\operatorname{dom}\sigma_{K_1})$ (see comments before Exercise 2.29 in [36, pp. 29-30]).

The claimed equality leads to the following: there exists $\varepsilon > 0$ such that $\partial_\varepsilon\sigma_{K_1}(z^*) \subseteq \mathbf{K}_2$. If not, we could construct a net $(z^{**}_\delta)_{\delta>0}$ such that $z^{**}_\delta \in \partial_\delta\sigma_{K_1}(z^*) \setminus \mathbf{K}_2$ for each $\delta > 0$. Let $z^{**} \in \mathbf{K_1}$ the $w^*$-limit of $(z^{**}_\delta)$ as $\delta \searrow 0$ (which exists since $z^* \in \operatorname{int}(\operatorname{dom}\sigma_{K_1})$ and therefore, $\partial_\delta\sigma_{K_1}(z^*)$ is $w^*$-compact for each $\delta > 0$). It is clear that

$$z^{**} \in \partial\sigma_{K_1}(z^*) = \partial\sigma_{K_2}(z^*) \subseteq \partial_{1/2}\sigma_{K_1}(x^*),$$

but, since $z^{**}_\delta \notin \mathbf{K}_2$, we get that $\langle z^{**}_\delta, x^*\rangle < \sigma_{K_1}(x^*) - 1$. Then, $\sigma_{K_1}(x^*) - \frac{1}{2} \leq \langle z^{**}, x^*\rangle \leq \sigma_{K_1}(x^*) - 1$, which is not possible.

Thus, there exists $\varepsilon > 0$ such that $\partial_\delta\sigma_{K_1}(z^*) = \partial_\delta\sigma_{K_2}(z^*)$ for each $\delta \leq \varepsilon$, and so, since $\eth\sigma_{K_2}(\cdot, z^*)$ is $\tau_0$-$w$ H-usc at $z^*$, we conclude that $\eth\sigma_{K_1}(\cdot, z^*)$ is also $\tau_0$-$w$ H-usc at $z^*$. By Proposition 3.2.8 and Proposition 3.1.4, we conclude that $z^* \in E[K_1, w] \cap \operatorname{int}(\operatorname{dom}\sigma_{K_1})$, which finishes the proof.

$\square$

**Corollary 3.2.11** *If $X$ has the FRNP, then every convex bounded closed set $K$ of $X$ is the closed convex hull of its weakly exposed faces, namely,*

$$K = \overline{\mathrm{co}}\left[\bigcup_{x^*\in E[K,w]} F[K, x^*]\right].$$

## 3.3 Strong subdifferentiability

In what follows, we will study the relation between a stronger version of the FRNP and a notion that has been around the literature for a while, namely, the *strong subdifferentiability.* This notion is a natural weakening of Fréchet-differentiability: We allow the subdifferential not to be a singleton but we keep the uniform convergence in the following sense:

**Definition 3.3.1** (Strong Subdifferentiability) *A function $f \in \Gamma_0(X)$ is said to be* strongly subdifferentiable (SSD) *at a point $x \in \operatorname{dom} f$ if the limit*

$$f'(x; h) = \lim_{t\searrow 0^+} \frac{f(x + th) - f(x)}{t}$$

*exists uniformly with respect to $h \in \mathbb{S}_X$.*

**Remark 3.3.2** Strong subdifferentiability differs from Fréchet-differentiability. Indeed, note that any continuous sublinear function is SSD at the origin. This can be easily deduce from the latter definition, using Propositions 1.2.26 and 1.2.27.

The strong subdifferentiability has been introduced in renorming theory (see, e.g., [20, pp. 88]), and it has been studied in relation with Fréchet-differentiable renormings, Asplund spaces, polyhedral spaces, proximinality and its relation with Hausdorff-upper semicontinuity (see [13], [15], [27]–[29]). Here, we will summarize the most relevant results about SSD.

**Proposition 3.3.3** ([13, Theorem 5.13]) *Let $f \in \Gamma_0(X^*, w^*)$ and let $x^* \in \operatorname{int}(\operatorname{dom} f)$. Then*

$$f \text{ is SSD at } x^* \iff \eth f(0,\cdot) \text{ is } \tau_{\|\cdot\|}\text{-}\tau_{\|\cdot\|} \text{ H-usc at } x^*.$$

**Theorem 3.3.4** (Godefroy, Montesinos and Zizler, [29, Theorem 1]) *Any separable Banach space with nonseparable dual admits an equivalent norm that is nowhere SSD except at the origin.*

**Theorem 3.3.5** (Godefroy, Montesinos and Zizler, [29, Theorem 2]) *If $X$ is a Banach space admitting an equivalent SSD norm (that is, SSD at every point of $X$), then $X$ is an Asplund space.*

An interesting example related to strong-subdifferentiability is the notion of polyhedral spaces, namely, Banach spaces for which their unit ball is a "polyhedron". Many notions of polyhedrality in infinite-dimensional spaces have been introduced in order to properly define a polyhedral space. We refer the reader to the work of R. Durier and P. L. Papini [22], in which the authors compare the different definitions of polyhedral space. Here, we present two of them: *Quasi-Polyhedrality* and *Polyhedral norm according to Klee.*

**Definition 3.3.6** (Polyhedral and Quasi-Polyhedral spaces) *Let $X$ be a Banach space. We say that*

*(a) A function $f \in \Gamma_0(X)$ is* quasi-polyhedral at $x \in \operatorname{dom} f$ *if there exists a neighborhood $V \in \mathcal{N}_X(x)$ such that*

$$\partial f(y) \subseteq \partial f(x), \quad \forall y \in V.$$

*$X$ is said to be a Quasi-Polyhedral space if its norm is quasi-polyhedral at each point of $\mathbb{S}_X$.*

*(b) The norm of $X$ is polyhedral (according to Klee) if for every finite subspace $Z$ of $X$, the set $Z \cap \mathbb{B}_X$ is a polyhedron. In such a case, we say that $X$ is a polyhedral space.*

The next proposition contains the structural properties of polyhedral and quasi-polyhedral spaces that we are interested in. The first one can be found in [22] (and the references therein). The second one is due to V. Fonf [25].

**Proposition 3.3.7** *Let $X$ be a Banach space. Then,*

*(a) If $X$ is quasi-polyhedral, then it is polyhedral.*

*(b) If $X$ is polyhedral, then it contains a subspace isomorphic to $c_0$.*

We finish this section with the following result which connects quasi-polyhedrality with strong subdifferentiability (which was stated for quasi-polyhedral norms in [27, Lemma 3.3]).

**Corollary 3.3.8** *Let $f \in \Gamma_0(X^*, w^*)$ and $x^* \in \operatorname{dom} f$. If $f$ is quasi-polyhedral at $x^*$, then $f$ is SSD at $x^*$.*

*Proof.* Since $f$ is quasi-polyhedral at $x^*$, then there exists a neighborhood $U$ of $x^*$ such that

$$\partial f(U) \subseteq \partial f(x^*).$$

Then, for every neighborhood $V$ in $\mathcal{N}_X(0)$, we have that

$$\eth f(0, y) \subseteq \eth f(0, x) \subseteq \eth f(0, x) + V, \qquad \forall y \in U.$$

By Proposition 3.3.3 and the equivalences of Proposition 3.1.4, the conclusion follows. □

## 3.4 Strong-FRNP

Taking Theorem 3.1.5 into account, let us consider the following natural strong version of property $(\mathcal{P})$ described in section 3.2, which we will denote as the property $(s\mathcal{P})$, defined as follows: For each Banach space $X$, the domain of $s\mathcal{P}_X$ is $D(X) = \Gamma_0(X^*, w^*) \times X^*$ and $s\mathcal{P}_X$ is given by

$$s\mathcal{P}_X(f, x^*) = 1 \iff x^* \in \operatorname{int}(\operatorname{dom} f^*) \text{ and } \eth f(0, \cdot) \text{ is } \tau_{\|\cdot\|}\text{-}\tau_{\|\cdot\|} \text{ H-usc at } x^*. \tag{3.4}$$

By Theorem 3.1.5, $(s\mathcal{P}) \implies (\mathcal{P})$.

**Proposition 3.4.1** *The property $(s\mathcal{P})$ is a convex $w^*$-smooth-like property.*

*Proof.* If we consider $\theta = \tau_{\|\cdot\|}$, Lemmas 3.2.1, 3.2.2 and 3.2.3 show that $(s\mathcal{P})$ is local, set-consistent and epigraphical, respectively. Also, Lemma 3.2.4 proves that $(s\mathcal{P})$ is $w^*$-transitive since for any two Banach spaces $X$ and $Y$ and any one-to-one bounded linear operator $T : Y \to X$, we have that $T$ is $\tau_{\|\cdot\|}\text{-}\tau_{\|\cdot\|}\big|_{T(Y)}$-open, according to the Banach open mapping theorem. □

So, coming back to Definition 1.3.17 and its adaptation to Banach spaces given at the beginning of this chapter, we naturally define strong-SDPD spaces and the strong-FRNP as follows:

**Definition 3.4.2** *A Banach space $X$ is said*

1. *to be a **strong-SDPD space** if it is $(s\mathcal{P})$-$w^*$-structural.*
2. *to have the **strong-FRNP** if it is $(s\mathcal{P})$-geometrical.*

As in the proof of Proposition 3.2.8, we can show that for a closed convex set $K \subseteq X$, $s\mathcal{P}[K] = E[K, \tau_{\|\cdot\|}]$, and so, we can use the results on [13] (see Proposition 3.1.4 and Proposition 3.3.3 above and [13, Theorem 5.13]) to characterize the strong-FRNP as follows:

**Proposition 3.4.3** *The following assertions are equivalent:*

*(a) For each convex closed set $K$ of $X$, $\sigma_K$ is SSD in a dense set of* $\operatorname{int}(\operatorname{dom}\sigma_K)$.

*(b) $X$ has the strong-FRNP.*

*(c) For every function $f \in \Gamma_0(X^*, w^*)$, there exists a dense set $D$ of* $\operatorname{int}(\operatorname{dom} f)$ *such that for each $x^* \in D$ and each sequence $(x_n^*, x_n) \in \operatorname{gph}(\eth f(0,\cdot))$ such that $x_n^* \to x^*$, we have that*

$$d(x_n, \eth f(0, x^*)) \to 0.$$

**Example 3.4.4** The unit ball of $X = c_0$ with the usual norm satisfies $E[\mathbb{B}_{c_0}, \tau_{\|\cdot\|}]$ is dense in $X^* = \ell^1$.

*Proof.* Recall that for a sequence $(x_n) \subseteq \mathbb{R}$ the support of $(x_n)$ is the set

$$\overline{\operatorname{supp}}[(x_n)] := \{n \in \mathbb{N} \ : \ x_n \neq 0\}. \tag{3.5}$$

Also recall that $c_{00}$ stands for the vector space given by all the real-valued sequences with finite support, which is known to be dense in $\ell^1$. Now, it is known (see [36, Example 14.b]) that for each $x^* = (x_n^*) \in \ell^1$,

$$\begin{aligned}\partial\|\cdot\|_1(x^*) &= \{x^{**} = (x_n^{**}) \in \ell^\infty \ : \ \forall n \in \overline{\operatorname{supp}}[x^*], \ x_n^{**} = \operatorname{sgn}(x_n^*)\} \\ &= \prod_{n\in\mathbb{N}} A_n,\end{aligned}$$

where $A_n = \{\operatorname{sgn}(x_n^*)\}$ if $n \in \overline{\operatorname{supp}}[x^*]$ and $A_n = [-1, 1]$, otherwise. In particular, it is clear that $x^*$ is a support functional of $\mathbb{B}_{c_0}$ if and only if $x^* \in c_{00}$, and in such a case, $x^* \in E[\mathbb{B}_{c_0}, w]$. We claim that in fact $c_{00} = E[\mathbb{B}_{c_0}, \tau_{\|\cdot\|}]$. Fix $x^* \in c_{00}$ with $\|x^*\|_1 = 1$, and define

$$\delta = \min_{k\in\overline{\operatorname{supp}}[x^*]} |x_k^*|,$$

so clearly $\delta > 0$. Fix $\varepsilon \in (0, \delta)$ and $x \in c_0 \cap \partial_{\delta\varepsilon} \|\cdot\|_1(x^*)$. Let $z \in c_0$ given by

$$z_n = \begin{cases} \operatorname{sgn}(x_n^*) & \text{if } n \in \overline{\operatorname{supp}}[x^*] \\ x_n & \text{if } n \in \mathbb{N} \setminus \overline{\operatorname{supp}}[x^*]. \end{cases}$$

It is clear that $z \in c_0 \cap \partial \|\cdot\|_1(x^*)$. Note also that for each $n \in \overline{\operatorname{supp}}[x^*]$, $x_n^* \cdot (z_n - x_n) = |x_n^*||z_n - x_n|$: Indeed, if $z_n = 1$, then $z_n - x_n \in [0, 2]$, and if $z_n = -1$, then $z_n - x_n \in [-2, 0]$. Thus, we have that

$$\delta\varepsilon \geq \sum_{n \in \overline{\operatorname{supp}}[x^*]} x_n^*(z_n - x_n) = \sum_{n \in \overline{\operatorname{supp}}[x^*]} |x_n^*||z_n - x_n| \geq \sum_{n \in \overline{\operatorname{supp}}[x^*]} \delta|z_n - x_n| \geq \delta\|z - x\|_\infty.$$

Then, $d(x, c_0 \cap \partial \|\cdot\|_1(x^*)) \leq \|z - x\|_\infty \leq \varepsilon$. Then, for each $\varepsilon \in (0, \delta)$,

$$\eth \|\cdot\|_1(\delta\varepsilon, x^*) \subseteq \eth \|\cdot\|_1(0, x^*) + \varepsilon\mathbb{B}_{c_0},$$

and so $x^* \in E[\mathbb{B}_{c_0}, \tau_{\|\cdot\|}]$. $\square$

A simpler proof of Example 3.4.4 follows just noting that the norm $\|\cdot\|_1$ of $\ell^1$ is quasi-polyhedral at each point $x^* \in c_{00}$, and so applying Proposition 3.3.8.

By small modifications in their proofs (just replace the weak topology by the strong topology and $E[K, w]$ by $E[K, \tau_{\|\cdot\|}]$), Proposition 3.2.10 and Corollary 3.2.11 remain true if we replace the FRNP by the strong-FRNP and $E[K, w]$ by $E[K, \tau_{\|\cdot\|}]$ in both statements.

Observe that property $(s\mathcal{P})$ satisfies the $w^*$-sum rule. Indeed, the $w^*$-sum rule follows immediately from the next lemma, which is simply a direct application of Proposition 1.2.27.

**Lemma 3.4.5** *Let $f, g \in \Gamma_0(X)$ and let $x \in \operatorname{int}(\operatorname{dom}(f + g))$. The function $f + g$ is SSD at $x$ if and only if both functions $f$ and $g$ are SSD at $x$.*

*Proof.* Since $x \in \operatorname{int}(\operatorname{dom}(f + g)) = \operatorname{int}(\operatorname{dom} f) \cap \operatorname{int}(\operatorname{dom} g)$, we know that for every $y \in \operatorname{dom}(f+g)$ we can write $\partial(f+g)(y) = \partial f(y) + \partial g(y)$. In particular, using Proposition 1.2.27(b), we have that $(f + g)'(x; \cdot) = f'(x; \cdot) + g'(x; \cdot)$, which allows us to write

$$\begin{aligned} &\limsup_{t \searrow 0} \sup_{h \in \mathbb{B}_X} \left( \frac{(f + g)(x + th) - (f + g)(x)}{t} - (f + g)'(x; h) \right) \\ \leq &\limsup_{t \searrow 0} \sup_{h \in \mathbb{B}_X} \left( \frac{f(x + th) - f(x)}{t} - f'(x; h) \right) + \limsup_{t \searrow 0} \sup_{h \in \mathbb{B}_X} \left( \frac{g(x + th) - g(x)}{t} - g'(x; h) \right). \end{aligned}$$

This yields the sufficiency of our statement. To prove the necessity, assume that $g$ is not SSD at $x$. We need to show that $f + g$ is not SSD at $x$. Since $g$ is not SSD, then there exist $\varepsilon > 0$, a sequence $(t_n) \subseteq ]0, +\infty[$ converging to 0 and a sequence $(h_n) \subseteq \mathbb{B}_X$ such that

$$\frac{g(x + t_n h_n) - g(x)}{t} - g'(x; h_n) > \varepsilon, \quad \forall n \in \mathbb{N}.$$

Then, using that $(f+g)'(x;\cdot) = f'(x;\cdot) + g'(x;\cdot)$ and Proposition 1.2.27(a), we can write for all $n \in \mathbb{N}$

$$\frac{(f+g)(x+t_n h_n) - (f+g)(x)}{t_n} - (f+g)'(x;h_n) \geq \frac{g(x+t_n h_n) - g(x)}{t} - g'(x;h_n) > \varepsilon,$$

and so, we get that

$$\limsup_{t\searrow 0} \sup_{h\in\mathbb{B}_X} \frac{(f+g)(x+th) - (f+g)(x)}{t} - (f+g)'(x;h) > \varepsilon.$$

This finishes the proof. $\square$

Now, applying Theorem 2.3.25 we can state the following improvement of Corollary 3.2.11:

**Corollary 3.4.6** *If $X$ has the strong-FRNP, then every convex bounded closed set $K$ of $X$ is the closed convex hull of its weakly exposed faces, namely,*

$$K = \overline{\text{co}} \left[ \bigcup_{x^*\in E[K,w]} F[K,x^*] \right].$$

*Moreover, if $X$ is separable, then the converse also holds true.*

## 3.5 Relations between the RNP and the FRNP

**Proposition 3.5.1** *Consider the following assertions:*

*(i) $X$ has the RNP.*

*(ii) $X$ has the strong-FRNP.*

*(iii) $X$ has the FRNP.*

*Then, (i)$\Rightarrow$(ii)$\Rightarrow$(iii).*

*Proof.* Clearly, for a function $f \in \Gamma_0(X^*, w^*)$ and a point $x^* \in \text{Int}[\text{dom} f]$,

$$f \text{ is F-differentiable at } x^* \Rightarrow f \text{ is SSD at } x^* \Rightarrow f'(x^*;\cdot) \text{ is } w^*\text{-lsc},$$

where the last implication comes from the fact that if $\eth f(0,\cdot)$ is $\tau_{\|\cdot\|}$-$\tau_{\|\cdot\|}$ H-usc at $x^*$, then is $\tau_{\|\cdot\|}$-$w$ H-usc at $x^*$ and an easy application of Theorem 3.1.5. Then, if we replace $f$ by any support function, the implications are direct. $\square$

The latter proposition presents the natural inclusions concerning RNP-like properties:

$$\text{RNP spaces} \subseteq \text{strong-FRNP spaces} \subseteq \text{FRNP spaces} \subseteq \text{Banach spaces.}$$

In the following, we will present some necessary conditions to have the FRNP or the strong-FRNP. In fact, we will show that not every Banach space has the FRNP, showing that the last inclusion is strict. We will close this chapter with a brief discussion about the main open question of this part of the thesis: Is the RNP equivalent or not with the FRNP or with the strong-FRNP?

### 3.5.1 Banach spaces without the FRNP/strong-FRNP

**Lemma 3.5.2** *Let $X$ be a Banach space with separable dual $X^*$. Then, for each equivalent norm $p$ of $X$ and each countable dense set $\{x_i^*\}_{i\in\mathbb{N}}$ of $\mathbb{B}_{(X^*,p_*)}$ the function $\rho : X^{**} \to \mathbb{R}_+$ given by*

$$\rho(x^{**}) = p_{**}(x^{**}) + \left(\sum_{i=1}^{\infty} 2^{-i}\langle x_i^*, x^{**}\rangle^2\right)^{\frac{1}{2}}$$

*is an equivalent rotund bidual norm of $X^{**}$.*

*Proof.* Clearly $\rho$ is an equivalent norm of $X^{**}$. Thus, in order to conclude, it is sufficient to show that

$$\left(\rho\big|_X\right)^{**} = \rho.$$

Consider the functions $r, r_n : X^{**} \to \mathbb{R}_+$ given by

$$r(x^{**}) = \left(\sum_{i=1}^{\infty} 2^{-i}\langle x_i^*, x^{**}\rangle^2\right)^{\frac{1}{2}}, \quad \text{and} \quad r_n(x^{**}) = \left(\sum_{i=1}^{n} 2^{-i}\langle x_i^*, x^{**}\rangle^2\right)^{\frac{1}{2}}.$$

It is clear that $\sup_n r_n = r$ and that for each $n \in \mathbb{N}$, $(r_n\big|_X)^{**} = r_n$. Now, fix $x^{**} \in X^{**}$ and $\delta > 0$. By construction, there exists $n_0 \in \mathbb{N}$ such that $r(x^{**}) \le r_{n_0}(x^{**}) + \delta$. Therefore, we can write

$$\begin{aligned} r(x^{**}) \le r_{n_0}(x^{**}) + \delta &= \left(\liminf_{X\ni x \xrightarrow{w^{**}} x^{**}} r_{n_0}(x)\right) + \delta \\ &\le \left(\liminf_{X\ni x \xrightarrow{w^{**}} x^{**}} r(x)\right) + \delta = \left(r\big|_X\right)^{**}(x^{**}) + \delta. \end{aligned}$$

Since $\delta$ and $x^{**}$ are arbitrary, we conclude that $r \le \left(r\big|_X\right)^{**}$, and so, since the other inequality always holds, we deduce that $r = \left(r\big|_X\right)^{**}$. Now, since $p$ and $r\big|_X$ are both continuous, by Proposition 1.2.11(c) we have that

$$(p + r\big|_X)^* = p_* \square (r\big|_X)^*,$$

and so, $\rho = p_{**} + r = (p + r\big|_X)^{**}$, which finishes the proof. $\square$

Recall that an equivalent norm $p$ on $X$ is said to be *rough* if there exists $\varepsilon > 0$ such that

$$\limsup_{p(h)\to 0} \frac{p(x+h)+p(x-h)-2p(x)}{p(x)} \geq \varepsilon, \quad \forall x \in X. \tag{3.6}$$

By Proposition 2.1.3, is easy to see that a rough norm is nowhere Fréchet-differentiable.

**Theorem 3.5.3** *Let $X$ be a Banach space satisfying*

*(a) $X^*$ is separable.*

*(b) There exists $K$ a nonempty convex, closed and bounded set of $X$ such that every relatively $w$-open set $W$ of $K$ has diameter* $\operatorname{diam}(W) > \varepsilon$*, for some $\varepsilon > 0$.*

*Then there exists an equivalent norm $p$ on $X$, such that the dual norm $p_*$ is Gâteaux-differentiable at every nonzero point and nowhere Fréchet-differentiable.*

*Proof.* Let $\tilde{K} = \overline{K}^{w^{**}} \subseteq X^{**}$. It is easy to see that, if $W$ is a relatively $w^{**}$-open set of $\tilde{K}$, then $W \cap K$ is nonempty, and therefore $W \cap K$ is a relatively $w$-open set of $K$. Then, by (b) we get

$$\operatorname{diam}(W) \geq \operatorname{diam}(W \cap K) > \varepsilon.$$

Let $C = \mathbb{B}_X + K + (-K)$ and $\tilde{C} = \mathbb{B}_{X^{**}} + \tilde{K} + (-\tilde{K})$. Let $p_{**}$ the norm in $X^{**}$ such that $\mathbb{B}_{(X^{**},p_{**})} = \tilde{C}$. Since $\tilde{C}$ is $w^{**}$-compact, $p_{**}$ is the dual norm of some equivalent norm $p_*$ on $X^*$. Let $\{x_i^* : i \in \mathbb{N}\}$ a dense set of $\mathbb{B}_{(X^*,p_*)}$, and define the function $\rho : X^{**} \to \mathbb{R}_+$ given by

$$\rho(x^{**}) = p_{**}(x^{**}) + \left( \sum_{i=1}^{\infty} 2^{-i} \langle x_i^*, x^{**} \rangle^2 \right)^{\frac{1}{2}}.$$

In the proof of [20, Theorem III.1.9] it is shown that $\rho$ is also a dual norm, and its predual norm is Gâteaux-differentiable at each nonzero point but which also is rough (and so, nowhere Fréchet-differentiable). Also, by Lemma 3.5.2, the predual norm of $\rho$ is also a dual norm in $X^*$, and therefore, the proof is finished. □

**Remark 3.5.4** The construction of the set $C = \mathbb{B}_X + (K) + (-K)$ shows that if there is a set $K$ which satisfies the condition (b) of Theorem 3.5.3, then there exists an equivalent norm $p$ over $X$ such that $\mathbb{B}_{(X,p)}$ also satisfies the same condition. It is known (see, e.g., [32, Theorem 3.1 and Corollary 3.2]), that condition (b) is equivalent to say that $X$ lacks the *Convex Point of Continuity Property* (CPCP).

Recall that $X$ has the (Convex) Point of Continuity Property if for every set $C$ (convex) bounded and $w$-closed, there exists $x \in C$ such that the identity mapping $\mathrm{id} : (C, w) \to (C, \tau_{\|\cdot\|})$ is continuous at $x$.

**Corollary 3.5.5** *Let $X$ be a Banach space with separable dual. If $X$ has the strong-FRNP, then it has the CPCP.*

*Proof.* Let us suppose that $X$ lacks the CPCP. By Theorem 3.5.3, there exists an equivalent norm $p$ on $X$ such that the dual norm $p_*$ on $X^*$ is Gâteaux-differentiable at each nonzero point and nowhere Frechét-differentiable.

Let $x^* \in \mathbb{S}_{(X^*,p_*)}$ with the Gâteaux derivative $D_G p_*(x^*) = \langle x, \cdot \rangle$ for some $x \in \mathbb{S}_{(X,p)}$ (such a point exists, by Proposition 2.1.7). It is clear that $F[\mathbb{B}_{(X,p)}, x^*] = \{x\}$. Then $x$ is an exposed point of $\mathbb{B}_{(X,p)}$, exposed by $x^*$. But, since $x$ is not strongly-exposed (otherwise, $p_*$ would be Fréchet-differentiable at $x^*$), there exists a sequence $(x_n) \subseteq \mathbb{B}_{(X,p)}$ and $\varepsilon > 0$ such that

$$\langle x^*, x_n \rangle \to \langle x^*, x \rangle, \quad \text{and} \quad p(x - x_n) \geq \varepsilon.$$

Then, for each $\alpha > 0$, there exists $n \in \mathbb{N}$ such $x_n \in S(\mathbb{B}_{(X,p)}, x^*, \alpha)$ and so,

$$S(\mathbb{B}_{(X,p)}, x^*, \alpha) \not\subseteq x + \frac{\varepsilon}{2}\mathbb{B}_{(X,p)} = F[\mathbb{B}_{(X,p)}, x^*] + \frac{\varepsilon}{2}\mathbb{B}_{(X,p)}.$$

Since $\alpha$ is arbitrary, we get that $F[\mathbb{B}_{(X,p)}, x^*]$ is not a strong-exposed face. Since $x^*$ is an arbitrary support functional of $\mathbb{B}_{(X,p)}$, we get that $E[\mathbb{B}_{(X,p)}, \tau_{\|\cdot\|}] = \emptyset$, which is a contradiction. $\square$

Now we will show that every Banach space containing an isomorphic copy of $c_0$ lacks the FRNP. To do so, we will use a theorem due to P. Morris [35], concerning to disappearance of extreme points:

**Theorem 3.5.6** (Morris, 1983) *Let $X$ be a separable Banach space containing an isomorphic copy of $c_0$. Then $X$ is isomorphic to a Banach space $Y$ endowed with a rotund norm $\|\cdot\|_Y$ such that*

$$\operatorname{ext}\left[\mathbb{B}_{(Y)}\right] \cap \operatorname{ext}\left[\mathbb{B}_{(Y^{**})}\right] = \emptyset.$$

**Proposition 3.5.7** *Let $X$ be a Banach space admitting a rotund equivalent norm $p$ on $X$ such that*

$$\operatorname{ext}\left[\mathbb{B}_{(X,p)}\right] \cap \operatorname{ext}\left[\mathbb{B}_{(X^{**},p_{**})}\right] = \emptyset.$$

*Then, $X$ lacks the FRNP.*

*Proof.* Let $x^*$ be a support functional of $\mathbb{B}_{(X,p)}$. Since $p$ is rotund, $x^*$ attains its norm at only one point $x \in \mathbb{B}_{(X,p)}$ and so, $\eth p_*(0, x^*)$ is the singleton $\{x\}$. If $x^* \in E[\mathbb{B}_{(X,p)}, w]$, then

$$\partial p_*(x^*) = \overline{X \cap \partial p_*(x^*)}^{w^{**}} = \{x\}.$$

Thus, $x \in \exp\left[\mathbb{B}_{(X^{**},p_{**})}\right] \subseteq \operatorname{ext}\left[\mathbb{B}_{(X^{**},p_{**})}\right]$, which is clearly a contradiction. Since $x^*$ is an arbitrary support functional of $\mathbb{B}_{(X,p)}$), $E[\mathbb{B}_{(X,p)}, w]$ has to be empty, finishing the proof. $\square$

**Corollary 3.5.8** *If $X$ contains a copy of $c_0$, then $X$ lacks the FRNP.*

*Proof.* Let $Z$ be a closed subspace of $X$ isomorphic to $c_0$. By Theorem 3.5.6, $Z$ is also isomorphic to a Banach space $Y$ having an equivalent rotund norm satisfying the conditions of Proposition 3.5.7. Thus, $Y$ lacks the FRNP.

Now, since the property $(\mathcal{P})$ used to define the FRNP (see Proposition 3.2.8) is a convex $w^*$-smooth-like property, Proposition 2.3.22 shows that $Z$ lacks the FRNP, and therefore so does $X$. □

Since every polyhedral space contains a copy of $c_0$ (see Proposition 3.3.7), we can state the following corollary:

**Corollary 3.5.9** *If a Banach space $X$ admits an equivalent polyhedral norm, then it lacks the FRNP.*

This last corollary is rather contradictory: Every quasi-polyhedral norm $p$ satisfies that $E[\mathbb{B}_{(X,p)}, \tau_{\|\cdot\|}]$ is dense in $X^*$, but the very existence of such a norm entails that the space is polyhedral and hence it lacks the FRNP.

### 3.5.2 Final Comments

It is not hard to see, due to the technology developed in Chapter 2 and Section 3.2 that, if we observe strong subdifferentiability in the primal space it induces a convex smooth-like property $(SSD)$: For each Banach space $X$, $D(X) := \Gamma_0(X) \times X$ and

$$SSD_X(f, x) = 1 \iff x \in \operatorname{int}(\operatorname{dom} f) \text{ and } f \text{ is SSD at } x. \tag{3.7}$$

Due to Theorem 3.3.4, we can write the following equivalence:

$X$ is separable and $(SSD)$-structural $\Leftrightarrow X$ is separable and Asplund $\Leftrightarrow X^*$ is separable,

but in the same paper where this theorem was presented (see [29]), Godefroy, Montesinos and Zizler asked whether or not this equivalence holds for nonseparable spaces. Since Asplund spaces are *separable reducible* (see Theorem 2.2.8), this question is equivalent to ask if $(SSD)$-structural spaces are stable for subspaces.

Stability for subspaces (resp. quotients) and for products (even with finite spaces) present somehow *the difficulty* of the convex smooth-like (resp. $w^*$-smooth-like) setting. We already encountered the same problem when we compared the RNP with the strong-FRNP and with the FRNP. Moreover, for these properties it seems to be even harder to prove equivalences, since we do not know the answer even in the separable setting.

These questions of stability seem to be in the heart of the classic theory of Banach spaces, and they are by no means easy questions. We definitely would like to come back to them in the future, and we think that it is a question of great interest.

# Bibliography - Part I


[1] M. Bachir and A. Daniilidis, "A dual characterisation of the Radon-Nikodým property", *Bulletin of the Australian Mathematical Society*, vol. 62, pp. 379–387, 2000.

[2] G. Beer, *Topologies on closed and closed convex sets.* Dordrecht Boston: Kluwer Academic Publishers, 1993.

[3] J. Benoist and A. Daniilidis, "Integration of Fenchel subdifferentials of epi-pointed functions", *SIAM Journal on Optimization*, vol. 12, no. 3, pp. 575–582, 2002.

[4] J. Borwein and D. Preiss, "A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions", *Transactions of the American Mathematical Society*, vol. 303, no. 2, pp. 517–527, 1987.

[5] J. Borwein and J. Vanderwerff, *Convex functions: constructions, characterizations and counterexamples.* Cambridge, UK New York: Cambridge University Press, 2010.

[6] N. Bourbaki, *Éléments de Mathématique: Algèbre (Chapitres 1 à 3).* Berlin New York: Springer, 2007.

[7] ——, *Éléments de Mathématique: Espaces Vectoriels Topologiques (Chapitres 1 à 5).* Berlin New York: Springer, 2007.

[8] ——, *Éléments de Mathématique: Topologie Générale (Chapitres 1 à 4).* Berlin New York: Springer, 2007.

[9] R. Bourgin, *Geometric Aspects of Convex Sets with the Radon-Nikodým Property.* New York: Springer-Verlag, 1983.

[10] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations.* New York London: Springer, 2011.

[11] A. Brønsted and R. T. Rockafellar, "On the subdifferentiability of convex functions", *Proceedings of the American Mathematical Society*, vol. 16, pp. 605–611, 1965.

[12] R. Burachik, J. E. Martinez-Legaz, and M. Rocco, “On a sufficient condition for equality of two maximal monotone operators”, *Set-Valued Analysis*, vol. 18, no. 3-4, pp. 327–335, 2010.

[13] A. K. Chakrabarty, P. Shunmunagaraj, and C. Zalinescu, “Continuity properties for the subdifferential and $\varepsilon$-subdifferential of a convex function and its conjugate”, *Journal of Convex Analysis*, vol. 14, no. 3, pp. 479–514, 2007.

[14] J. Collier, “The dual of a space with the Radon-Nikodým property”, *Pacific Journal of Mathematics*, vol. 64, no. 1, pp. 103–106, 1976.

[15] M. D. Contreras and R. Payá, “On upper semicontinuity of duality mappings”, *Proceedings of the American Mathematical Society*, vol. 121, no. 2, pp. 451–459, 1994.

[16] J. Conway, *A Course in Functional Analysis.* New York: Springer-Verlag, 1990.

[17] R. Correa, Y. Garcia, and A. Hantoute, “Integration formulas via the Fenchel subdifferential of nonconvex functions”, *Nonlinear Analysis*, vol. 75, no. 3, pp. 1188–1201, 2012.

[18] R. Correa, A. Hantoute, and D. Salas, “Integration of nonconvex epi-pointed functions in locally convex spaces”, *Journal of Convex Analysis*, vol. 23, pp. 511–530, 2016.

[19] R. Correa and A. Jofré, “Some properties of semismooth and regular functions in nonsmooth analysis”, in *Recent Advances in System Modelling and Optimization*, ser. Lecture Notes in Control and Information Sciences, L. Contesse, R. Correa, and A. Weintraub, Eds., vol. 87, Springer Berlin Heidelberg, 1986, pp. 69–85.

[20] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces.* Harlow, Essex, England New York, NY: Longman Scientific & Technical Wiley, 1993.

[21] J. Diestel and J. J. Uhl, *Vector measures.* Providence, R.I: American Mathematical Society, 1977.

[22] R. Durier and P. L. Papini, “Polyhedral norms in an infinite dimensional space”, *Rocky Mountain Journal of Mathematics*, vol. 23, no. 3, 863?875, 1993.

[23] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach Space Theory: The Basis for Linear and Nonlinear Analysis (CMS Books in Mathematics).* New York: Springer, 2010.

[24] M. Fabian and V. Zizler, "An elementary approach to some questions in higher order smoothness in Banach spaces", *Extracta Mathematicae*, vol. 14, no. 3, pp. 295–327, 1999.

[25] V. P. Fonf, "Polyhedral Banach spaces", *Mathematical Notes*, vol. 30, no. 3-4, pp. 809–813, 1981.

[26] J. R. Giles, "A subdifferential characterisation of Banach spaces with the Radon-Nikodým property", *Bulletin of the Australian Mathematical Society*, vol. 66, pp. 313–316, 2002.

[27] G. Godefroy and V. Indumathi, "Strong proximinality and polyhedral spaces", *Revista Matemática Coplutense*, vol. XIV, no. 1, pp. 105–125, 2001.

[28] ——, "Norm-to-Weak upper semi-continuity of the duality and pre-duality mappings", *Set-Valued Analysis*, vol. 10, pp. 317–330, 2002.

[29] G. Godefroy, V. Montesinos, and V. Zizler, "Strong subdifferentiability of norms and geometry of Banach spaces", *Commentationes Mathematicae Universitatis Carolinae*, vol. 36, no. 6, pp. 493–502, 1995.

[30] J. B. Hiriart-Urruty and J. Benoist, "What is the subdifferential of the closed convex hull of a function?", *SIAM Journal on Mathematical Analysis*, vol. 27, no. 6, pp. 1661–1679, 1996.

[31] J. E. Martinez-Legaz, "Integration of the Fenchel subdifferentials revisited", *Vietnam Journal of Mathematics*, vol. 42, no. 4, pp. 533–542, 2014.

[32] W. B. Moors, "The relationship between Goldstine's theorem and the convex point of continuity property", *Journal of Mathematical Analysis and Applications*, vol. 188, pp. 819–832, 1994.

[33] J. J. Moreau, "Proximité et dualité dans un espace hilbertien", *Bulletin de la Société Mathématique de France*, vol. 93, pp. 273–299, 1965.

[34] ——, "Fonctionnelles convexes", *Séminaire Jean Leray, Collège de France*, no. 2, pp. 1–108, 1966-1967.

[35] P. Morris, "Disappearance of extreme points", *Proceedings of the American Mathematical Society*, vol. 88, no. 2, pp. 244–246, 1983.

[36] R. R. Phelps, *Convex functions, monotone operators, and differentiability*. Berlin New York: Springer-Verlag, 1989.

[37] R. Poliquin, "Integration of subdifferentials of nonconvex functions", *Nonlinear Analysis: Theory, Methods and Applications*, vol. 17, no. 4, pp. 385–398, 1991.

[38] R. T. Rockafellar, "Characterization of the subdifferentials of convex functions", *Pacific Journal of Mathematics*, vol. 17, pp. 497–510, 1966.

[39] R. T. Rockafellar and R. Wets, *Variational analysis*. Berlin New York: Springer, 1998.

[40] D. Salas, "Convex smooth-like properties and Faces Radon-Nikodým property in Banach spaces", (Submitted).

[41] ——, "Fórmula de integración en espacios con la propiedad de continuidad del subdiferencial", Available at `http://tesis.uchile.cl/handle/2250/114677`, Master's thesis, Universidad de Chile (Facultad de Ciencias Físicas y Matemáticas), Santiago, Chile, 2013.

[42] H. Schaefer and M. Wolff, *Topological vector spaces*. New York: Springer, 1999.

[43] W. Schahermayer, "The Radon-Nikodým property and the Krein-Milman property are equivalent for strongly regular sets", *Transactions of the American Mathematical Society*, vol. 303, no. 2, pp. 673–687, 1987.

[44] C. Stegall, "The Radon-Nikodým property in conjugate Banach spaces", *Transactions of the American Mathematical Society*, vol. 206, pp. 213–223, 1975.

[45] ——, "The duality between Asplund spaces and spaces with the Radon-Nikodým property", *Israel Journal of Mathematics*, vol. 29, no. 4, pp. 408–412, 1978.

[46] L. Thibault and D. Zagrodny, "Integration of subdifferentials of lower semicontinuous functions on Banach spaces", *Journal of Mathematical Analysis and Applications*, vol. 189, no. 1, pp. 33–58, 1995.

[47] ——, "Subdifferential determination of essentially directionally smooth functions in Banach space", *SIAM Journal on Optimization*, vol. 20, no. 5, pp. 2300–2326, 2010.

[48] L. Thibault and N. Zlateva, "Integrability of subdifferentials of directionally Lipschitz functions", *Proceedings of the American Mathematical Society*, vol. 133, no. 10, pp. 2939–2948, 2005.

[49] L. Veselý, "Boundary of Polyhedral spaces: an alternative proof", *Extracta Mathematicae*, vol. 15, no. 1, pp. 213–217, 2000.

[50] K. Yosida and E. Hewitt, "Finitely additve measures", *Transactions of the American Mathematical Society*, vol. 72, no. 1, pp. 46–66, 1952.

[51] C. Zalinescu, *Convex analysis in general vector spaces.* River Edge, N.J. London: World Scientific, 2002.

# Part II

# Nonconvex Geometry: Smoothness of the metric projection in Hilbert spaces

# Introduction of Part II

In his 1973 fundamental paper [17], R. B. Holmes showed that, whenever we have a closed convex set $K$ in a Hilbert space $X$ such that

(i) $K$ has nonempty relative interior (namely, the interior of $K$ as a subset of $Y = \overline{\mathrm{aff}}(K)$ is nonempty), and

(ii) the boundary of $K$ as a subset of $Y$, $\mathrm{bd}\, K$, is a $\mathcal{C}^{p+1}$-submanifold at a point $x_0 \in \mathrm{bd}\, K$, where $p$ is a positive integer,

then the metric projection $P_K$ is a mapping of class $\mathcal{C}^p$ in an open neighborhood $W$ of the open normal ray

$$\mathrm{Ray}_{x_0}(K) := \{x_0 + t\nu \ : \ t > 0\},$$

where $\nu$ denotes the unit exterior normal vector of $K$ at $x_0$. The main steps of his approach to arrive to this theorem were:

1. It is enough to prove the theorem for convex bodies (namely, where $K$ has nonempty interior), since under (i), restricting to the case $0 \in K$ (after suitable translation) we can write

   $$P_K = \left(P_K\big|_Y\right) \circ \Pi_Y,$$

   where $\Pi_Y$ denotes the orthogonal projection to $Y$ (which is a continuous linear mapping and therefore of class $\mathcal{C}^\infty$);

2. The smoothness of $\mathrm{bd}\, K$ at $x_0$ can be translated as the smoothness of the Minkowski functional $\rho_K$ (independently of which translation is used to ensure that 0 is an interior point of $K$); furthermore, the equality $\nu = \|\nabla\rho_K(x_0)\|^{-1}\nabla\rho_K(x_0)$ holds true;

3. The distance function $d_K$ is of class $\mathcal{C}^1$ in $X \setminus K$; and finally,

4. For any point $x \in \mathrm{Ray}_{x_0}(K)$ and a suitable choice of neighborhoods $U$ and $V$ of $x$

and $x_0$ respectively, the mapping

$$\begin{aligned} F : U \times V &\to X \\ (u,v) &\mapsto u - v - d_K(u)\frac{\nabla \rho_K(v)}{\|\nabla \rho_K(v)\|} \end{aligned}$$

is well defined, of class $\mathcal{C}^1$, and for every $(u,v) \in U \times V$, one has

$$F(u,v) = 0 \iff v = P_K(u).$$

With all these features, Holmes concluded his theorem through an application of the well-known Implicit Function Theorem.

After Holmes' contribution, in 1982, S. Fitzpatrick and R. R. Phelps continued in [16] the study of smoothness of the metric projection onto convex bodies. They observed that, whenever a convex body $K$ had $\mathcal{C}^{p+1}$-smooth boundary, then for every $x \in X \setminus K$ the differential $DP_K(x)$ is a surjective mapping from $X$ onto $H[x]$ where

$$H[x] := \{h \in X \ : \ \langle h, x - P_K(x)\rangle = 0\},$$

and it is invertible as a mapping from $H[x]$ onto $H[x]$. This behavior of $DP_K(x)$ was implicitly obtained as a corollary in Holmes' proof. Using this necessary condition and also the equivalence between the smoothness of $\operatorname{bd} K$ and the smoothness of $\rho_K$, they were able to prove that a convex body $K$ has $\mathcal{C}^{p+1}$-smooth boundary if and only if the metric projection $P_K$ is of class $\mathcal{C}^p$ on $X \setminus K$ and for each $x \in X \setminus K$, $DP_K(x)$ is invertible as a mapping from $H[x]$ to $H[x]$.

In the same paper, they also give two counterexamples: The first consist in an example of a convex closed set $C_1$ for which its metric projection $P_K$ is of class $\mathcal{C}^1$ on $X \setminus C_1$ but $\operatorname{bd} C_1$ fails to be even a $\mathcal{C}^1$-submanifold. Here is the invertibility of $DP_{C_1}(\cdot)$ what fails. The second one, is a convex set $C_2$ with boundary $\mathcal{C}^{1,1}$ for which its metric projection $P_{C_2}$ is nowhere Fréchet-differentiable on $X \setminus C_2$. Here is the $\mathcal{C}^{p+1}$-smoothness of the boundary what doesn't hold. With those two results, they gave a very complete framework of the smoothness of the metric projection in the convex case.

Some other contributions have been made concerning to the differentiability properties of the metric projection on the convex setting, such as [18], on which J. Kruskal provided an example of a closed convex set in $\mathbb{R}^3$ such that its metric projection fails to have one-side directional derivative at infinitely many points, and [22], on which D. Noll explored other notions of differentiability for the metric projections onto convex sets.

Parallel, in 1984, J.-B. Poly and G. Raby studied the smoothness of the metric projection $P_M$ onto a $\mathcal{C}^{p+1}$-submanifold $M$ of a finite-dimensional space $\mathbb{R}^n$. They proved that a closed set $M \subseteq \mathbb{R}^n$ is a $\mathcal{C}^{p+1}$-submanifold at a point $m_0 \in M$ if and only if the square of the distance function $d_M^2(\cdot)$ is of class $\mathcal{C}^{p+1}$ near $m_0$. Their strategy consists in regard $M$

near $m_0$ as the graph of a mapping $\theta$ whose differential $D\theta$ can be expressed in terms of the metric projection $P_M$, which is well-defined near $m_0$ due to the local compactness of $M$.

Following the way opened by the strategy of Holmes, the aim of this second part of the thesis is to study, under the same hypotheses (i)-(ii), similar *local* results dropping the hypothesis of convexity and replacing it with *prox-regularity*, and, as Fitzpatrick and Phelps did, to provide a converse of this extension. We work with a family of sets called *closed bodies* which suitably extends the notion of convex body, used in the classic theory. Finally, through this work, we integrate the study of the case when the set onto which we are projecting is a submanifold itself, mixing up our results with those obtained by Poly and Raby. In order to do so, our key observation is that we can locally translate the smoothness of the boundary of a set to the smoothness of a suitable real-valued function (not necessarily unique) and therefore, compensate the loss of the Minkowski functional.

This part is mainly based on the joint work with R. Correa and L. Thibault [12], and the work in progress with L. THibault [28]. The main motivation for this research came from the huge advances made in Proximal Analysis and from the 2000's paper by Poliquin, Rockafellar and Thibault [24], which allows us to replace the continuous differentiability of the distance function to convex bodies, with another suitable one related to prox-regular sets. Also we want to mention Mazade Ph. D. Thesis [19], in which local prox-regularity was profoundly studied in a quantified sense, and Shapiro's paper [29] and Canino's paper [7], on which the directional differentiability of the metric projection was independently studied in the nonconvex framework, under the notions of $O(2)$-convexity and $p$-convexity, respectively. Nowadays, it is known that these two notions coincide with the notion of prox-regularity.

This part is organized as follows: Below this introduction, we present the basic notation we will use in through this part. After, Chapter 4 summarizes the theory of differential submanifolds (section 4.1), Proximal and Clarke's nonsmooth calculus(section 4.2), and the notions of epi-Lispchitzianity and prox-regularity of sets (section 4.3). All that content is known and had been extracted mainly from [3], [9]–[11], [23], [30]. Even though many of the notions and propositions presented are still valid in larger contexts (Uniformly Convex spaces or Banach spaces), we will restrict ourself to present the theory in the infinite-dimensional Hilbert setting. The chapter ends with section 4.4, on where we pose some useful variational properties of sets with smooth boundary (or which are submanifolds themselves), some of them in terms of prox-regularity and epi-Lipschitzianity. We remark Proposition 4.4.4, which will allows us to represent these sets locally as the epigraph of suitable functions, and Theorem 4.4.6, which is the direct extension of Poly and Raby's theorem in the infinite-dimensional setting.

Chapter 5 is the core of this work. In section 5.1 we present our main results extending Holmes' Theorem. We show in Theorem 5.1.6 that whenever a closed body has $\mathcal{C}^{p+1}$-smooth boundary, then its metric projection is of class $\mathcal{C}^p$ in a suitable quantified neighborhood of the set (section 5.1.1). We also prove in Theorem 5.1.10 the same results for submanifolds (section 5.1.2). In section 5.2, we follow the strategy of Fitzpatrick and Phelps, studying the behavior of the derivative of the metric projection when it exists and proving in Theorem 5.2.11 the converse of Theorem 5.1.6 and obtaining a full characterization of nonconvex bodies with smooth boundary in terms of the smoothness of their metric projections ( section 5.2.2). We finish this work with a partial converse of Theorem 5.1.10 following the strategy of Poly and Raby (section 5.2.3).

# Notation of Part II

In the following (Chapters 4 and 5), $X$ will always stand for a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and $\|\cdot\|_X$ (or simply $\|\cdot\|$ if there is no confusion) will be its Hilbert norm. Using Riesz's representation Theorem, we will identify the dual space of $X$ with $X$ itself. For a point $x \in X$ we will denote by $\mathcal{N}_X(x)$ (or simply by $\mathcal{N}(x)$ if there is no confusion) the set of all neighborhoods of $x$ for the norm-topology. By $B_X[x,\alpha]$ and $B_X(x,\alpha)$ we mean the closed and open ball centered in $x$ with radius $\alpha > 0$, respectively. We will also write $\mathbb{B}_X$ and $\mathbb{S}_X$ to denote the unit ball $B_X[0,1]$ and the unit sphere $B_X[0,1] \setminus B_X(0,1)$, respectively.

We will also write $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$. In the following, $p$ will stand for an integer larger or equal to 1. We will recall this assumption whenever is needed. For two points $x, y \in X$ we will write

$$[x,y] := \{tx + (1-t)y \ : \ t \in [0,1]\}.$$

For two closed subspaces $Y, Z$ of $X$ such that $X$ can be written as the direct sum of $Y$ and $Z$ (which is denoted by $X = Y \oplus Z$), we will denote by $\pi_{Y,Z}$ and $\pi_{Z,Y}$ (or simply $\pi_Y$ and $\pi_Z$ if there is no confusion) the parallel projections associated to such a decomposition. In the case that $X = Y \times Z$, we will write $\pi_Y$ and $\pi_Z$ instead of $\pi_{Y\times\{0\}}$ and $\pi_{Z\times\{0\}}$, respectively.

For a closed subspace $Z$ of $X$, we will denote by $Z^\perp$ its orthogonal subspace, that is,

$$Z^\perp = \{x \in X \ : \ \langle x, z\rangle = 0, \ \forall z \in Z\}.$$

It is known that $Z^\perp$ is a closed subspace of $X$ and that we can write $X = Z \oplus Z^\perp$. For $Z$, we will also use the notation $\Pi_Z$ to denote its orthogonal projection, namely, $\Pi_Z(x)$ is the only point of $Z$ such that

$$\|x - \Pi_Z(x)\| = \inf\{\|x - z\| \ : \ z \in Z\}.$$

It is known that $\Pi_Z \equiv \pi_{Z,Z^\perp}$.

For another Hilbert space $Y$, we will denote by $\mathcal{L}(X;Y)$ the space of all bounded linear operators from $X$ into $Y$, endowed with the operator norm which is given by

$$\|T\|_{\mathcal{L}(X;Y)} = \sup_{x\in\mathbb{S}_X} \|Tx\|_Y.$$

If $Y = X$, we will simply write $\mathcal{L}(X)$. We will denote by $\mathrm{id}_X$ the identity map from $X$ to $X$, which is an operator in $\mathcal{L}(X)$. Finally, for an operator $T \in \mathcal{L}(X;Y)$ we will denote by $T^*$ its adjoint operator, that is, the only operator in $\mathcal{L}(Y;X)$ satisfying

$$\langle y, Tx\rangle = \langle T^* y, x\rangle, \ \forall (x,y) \in X \times Y.$$

We say that an operator $T \in \mathcal{L}(X)$ is self-adjoint (or symmetric) if $T^* = T$. We also will write $\operatorname{Ker} T$ and $\operatorname{Im} T$ to denote the kernel and the image of $T$. Indistinctly, we will also write $TX$ instead of $\operatorname{Im} T$.

Recall that a linear operator $T \in \mathcal{L}(X)$ is an orthogonal projection (i.e., $T = \Pi_Z$ for some closed subspace $Z$ of $X$) if and only if it is idempotent (i.e., $T \circ T = T$) and symmetric. We refer the reader to [20, Ch. 11] for the proof of this fact in the finite-dimensional setting, which remains exactly the same for arbitrary Hilbert spaces.

For a mapping $F : U \subset X \to Y$ and $V \subseteq U$, we will write $F\big|_V$ the restriction of $F$ to $V$. We will also write $\operatorname{gph} F$ to denote its graph. If $Y = \mathbb{R}$ we will write $\operatorname{epi} F$, $\operatorname{hypo} F$, $\operatorname{epi}_s F$ and $\operatorname{hypo}_s F$ to denote its epigraph, hypograph, strict epigraph and strict hypograph, respectively, that is,

$$\begin{aligned}
\operatorname{epi} F &:= \{(u,r) \in U \times \mathbb{R} \ : \ F(u) \leq r\},\\
\operatorname{hypo} F &:= \{(u,r) \in U \times \mathbb{R} \ : \ F(u) \geq r\},\\
\operatorname{epi}_s F &:= \{(u,r) \in U \times \mathbb{R} \ : \ F(u) < r\},\\
\operatorname{hypo}_s F &:= \{(u,r) \in U \times \mathbb{R} \ : \ F(u) > r\}.
\end{aligned}$$

For a set $S \subseteq X$ we will write $\operatorname{int} S$, $\operatorname{cl} S$ and $\operatorname{bd} S$ to denote the interior, the closure and the boundary of $S$, respectively. If $U$ is a subset of $X$ containing $S$, we will write $\operatorname{int}_U S$, $\operatorname{cl}_U S$ and $\operatorname{bd}_U S$ to denote the interior, the closure and the boundary of $S$ relative to $U$, respectively. We will also use $\overline{S}$ and $\overline{S}^U$ instead of $\operatorname{cl} S$ and $\operatorname{cl}_U S$, indistinctly.

For a mapping $F : U \subseteq X \to Y$ (with $U$ a nonempty open set) we will say that $F$ is G-differentiable (resp. F-differentiable) at $x \in U$ if it is Gâteaux-differentiable (resp. Fréchet-differentiable) at $x$. We will denote by $D_G F(x)$ the Gâteaux-derivative (G-derivative, for short) of $F$ at $x$ and by $D_F F(x)$ or simply $DF(x)$ the Fréchet-derivative (F-derivative, for short) of $F$ at $x$.

In the special case that $Y = \mathbb{R}$, we will denote by $\nabla F(x)$ the gradient of $F$ at $x$, that is, the unique element of $X$ such that

$$D_G F(x) = \langle \nabla F(x), \cdot\rangle.$$

For an integer $p \geq 1$, if $F$ is $p$-times F-differentiable, then we will denote by $D^p F(x)$ its $p$th F-derivative. Let us recall the *little o* notation of differentiability. The mapping $F$ is F-differentiable at $x$ if and only if we can write

$$F(x+h) = F(x) + Ah + o(h)$$

for some $A \in \mathcal{L}(X;Y)$, where $o(h)$ is a mapping from $X$ to $Y$ satisfying $\|h\|^{-1}o(h) \to 0$ as $h \to 0$. In such a case, $A$ coincides with $DF(x)$.

Analogously, the mapping $F$ is G-differentiable at $x$ if and only if for every $h \in X$ we can write

$$F(x+th) = F(x) + tAh + o(t)$$

for some $A \in \mathcal{L}(X;Y)$, where $o(t)$ is a function from $]0,+\infty[$ to $Y$ satisfying $t^{-1}o(t) \to 0$ when $t \searrow 0$.

Recall that $F$ is said to be of class $\mathcal{C}^p$ (resp. $\mathcal{C}^{p,1}$) near $x$, if there exists a neighborhood $U \in \mathcal{N}_X(x)$ such that $F$ is $p$-times F-differentiable at each $u \in U$ and the $p$th derivative $D^pF(\cdot)$ is continuous (resp. locally Lipschitz-continuous) on $U$.

As introduced in the Part I, for a function $f : X \to \mathbb{R}_\infty$, we will denote by $\operatorname{dom} f$ and $\partial f(x)$ its effective domain and its (convex) subdifferential. Also, for a subset $S$ of $X$, $I_S$ denotes the *indicator function* of $S$, that is

$$I_S : X \to \mathbb{R}_\infty, \qquad x \mapsto I_S(x) = \begin{cases} 0 & x \in S \\ +\infty & x \notin S. \end{cases}$$

# Chapter 4

# Hilbertian manifolds and Prox-regular sets

In this chapter we will give a small overview of some results concerning differential manifolds modeled over an infinite-dimensional Hilbert space and its relationship with the theory of prox-regular sets. This summary is focused on the results that we will use in the study of smoothness of the metric projection onto nonconvex sets, and therefore, we will skip many classical results of the theory of infinite-dimensional differential manifolds.

## 4.1 Hilbertian $\mathcal{C}^p$-submanifolds

**Definition 4.1.1** ($\mathcal{C}^p$-submanifolds) *A subset $M$ of $X$ is said to be a* (Hilbertian) $\mathcal{C}^p$-submanifold *at a point $m_0 \in M$ if there exists an open neighborhood $U \in \mathcal{N}_X(m_0)$, a closed subspace $Z$ of $X$ (called the model space) and a mapping $\varphi : U \to \varphi(U) \subseteq X$ such that*

1. *$\varphi$ is a $\mathcal{C}^p$-diffeomorphism, that is, $\varphi(U)$ is an open set of $X$, $\varphi : U \to \varphi(U)$ is bijective and $\varphi, \varphi^{-1}$ are both mappings of class $\mathcal{C}^p$.*
2. *$\varphi(m_0) = 0$ and $\varphi(M \cap U) = Z \cap \varphi(U)$.*

*We simply say that $M$ is a $\mathcal{C}^p$-submanifold if it is so at each point $m \in M$ with the same model space $Z$.*

From the definition, it is clear that if $M$ is a $\mathcal{C}^p$-submanifold at $m_0$, then it is so at any point in a neighborhood of $m_0$ relative to $M$. Also, the pair $(U, \varphi)$ and the model space $Z$ need not to be unique in order to represent the submanifold $M$ at the point $m_0$. In fact, the possible pairs $(U, \varphi)$ are called *local charts at $m_0$* and each one describes $M$ as a submanifold at $m_0$. Furthermore, when we have two local charts $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ with two different model spaces $Z_1$ and $Z_2$ we get that both model spaces are isomorphic. Indeed, consider the mapping $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$. By chain rule, we have that $D(\varphi_2 \circ \varphi_1^{-1})(0) = D\varphi_2(m_0) \circ (D\varphi_1)^{-1}(0)$, and also it is not hard to prove that

$$D(\varphi_2 \circ \varphi_1^{-1})(0)Z_1 \subseteq Z_2.$$

Exchanging the roles of $\varphi_1$ and $\varphi_2$, we get that $D(\varphi_1 \circ \varphi_2^{-1})(0)Z_2 \subseteq Z_1$, and since

$$D(\varphi_1 \circ \varphi_2^{-1})(0) = (D(\varphi_2 \circ \varphi_1^{-1})(0))^{-1},$$

we conclude that $Z_1$ and $Z_2$ are isomorphic, as we claimed, where a linear isomorphism is given by $D(\varphi_2 \circ \varphi_1^{-1})(0)\big|_{Z_1}$.

When we have that $M$ is a submanifold at each point, we can choose a family $\mathcal{A} = \{(U_i, \varphi_i) \ : \ i \in I\}$ of local charts with the same model space $Z$ such that

1. $\{U_i \ : \ i \in I\}$ is an open cover of $M$.
2. Whenever $U_i \cap U_j \neq \emptyset$, the map $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is a $\mathcal{C}^p$-diffeomorphism.

These types of families over a submanifold are called *atlases* and describe completely the differential and topological structure of $M$. We won't use them in the development of this work, but atlases play a fundamental role in differential manifolds theory. We refer the reader to [1, Chapter 3] for further information.

We will now introduce the notion of *Tangent (vector) space* associated to a submanifold. Recall that a $\mathcal{C}^1$-curve $\gamma$ on a set $M \subseteq X$ is a differentiable function from an open interval $I \subset \mathbb{R}$ with values in $M$.

**Definition 4.1.2** (Tangent space) *Let $M \subseteq X$ be a $\mathcal{C}^p$-submanifold at a point $m_0 \in M$. We define the* Tangent (vector) space *of $M$ at $m_0$ as the set*

$$T_{m_0}M := \left\{h \in X \ : \ \exists \gamma : ]-1,1[\to M,\ \mathcal{C}^1\text{-curve with } \gamma(0) = m_0 \text{ and } \gamma'(0) = h\right\}.$$

It is direct from the definition that $T_{m_0}M$ is a closed subspace of $X$ and that for every local chart $(U, \varphi)$ at $m_0$ with model space $Z$, we have that

$$D\varphi^{-1}(0)Z = T_{m_0}M. \tag{4.1}$$

Indeed, if $h \in T_{m_0}M$, there exists a $\mathcal{C}^1$-curve $\gamma : ]-1,1[\to M$ such that $\gamma(0) = m_0$ and $\gamma'(0) = h$. We may suppose that the curve is contained in $U$ and so, we have that the mapping $\tilde{\gamma} : ]-1,1[\to Z$ given by $\tilde{\gamma}(t) = \varphi \circ \gamma(t)$ is a $\mathcal{C}^1$-curve in $Z$. Then,

$$h = \gamma'(0) = (D\varphi(m_0))^{-1} \circ \tilde{\gamma}'(0) = D\varphi^{-1}(0) \circ \tilde{\gamma}'(0) \in D\varphi^{-1}(0)Z.$$

For the other inclusion, fix $z \in Z$ and consider the mapping $\gamma : ]-1,1[ \to M$ given by

$$\gamma(t) = \varphi^{-1}(t\varepsilon z),$$

where $\varepsilon > 0$ is small enough such that the open segment $(-\varepsilon z, \varepsilon z) \subset \varphi(U)$. Clearly, $\gamma$ is a $\mathcal{C}^1$-curve and $\gamma(0) = m_0$. Therefore, $\gamma'(0) = \varepsilon D\varphi^{-1}(0)z \in T_{m_0}M$. Finally, since $T_{m_0}M$ is a vector space, we get that $D\varphi^{-1}(0)z \in T_{m_0}M$, which proves (4.1).

**Example 4.1.3** Let $Z$ be a subspace of $X$ and $Y$ be another Hilbert space. Then,

(a) Every open set $V$ of $Z$ is a $\mathcal{C}^\infty$-submanifold of $X$. It is enough to consider the local chart given by $(U, \mathrm{id}_X)$, where $U$ is any open set of $X$ with $U \cap Z = V$.

(b) (Immersion Theorem; see [1, Theorem 3.5.8]) Let $V$ be an open subset of $Z$ and $r : V \to X$ be a $\mathcal{C}^p$-*embedding*, that is,

(i) $r$ is an injective mapping of class $\mathcal{C}^p$;

(ii) at each $v \in V$, $Dr(v)$ is injective with closed range; and

(iii) $r$ is a homeomorphism between $V$ and $r(V)$.

Then, $r(V)$ is a $\mathcal{C}^p$-submanifold with model space $Z$ and for every $v \in V$, $T_{r(v)}r(V) = Dr(v)Z$.

(c) Let $U$ be an open set of $X$ and $f : U \to Y$ be a function of class $\mathcal{C}^p$. Then $\mathrm{gph}\, f$ is a $\mathcal{C}^p$-submanifold of $X \times Y$ and, for every $u \in U$, the Tangent space $T_{(u,f(u))}(\mathrm{gph}\, f)$ coincides with $\mathrm{gph}\, Df(u)$, and so is isomorphic to $X$. To prove this, it is enough to apply the Immersion Theorem to the $\mathcal{C}^p$-embedding $r : U \to X \times Y$ given by $r(u) = (u, f(u))$.

(d) If $M \subseteq X$ is a $\mathcal{C}^p$-submanifold and $\varphi : X \to Y$ is a $\mathcal{C}^p$-diffeomorphism, then $\varphi(M)$ is a $\mathcal{C}^p$-submanifold of $Y$. In such a case, for each $m \in M$ we have that $T_{\varphi(M)}\varphi(m) = D\varphi(m)T_mM$.

The first proposition we will show is that every $\mathcal{C}^p$-submanifold can be represented as the graph of a $\mathcal{C}^p$-mapping.

**Proposition 4.1.4** *Let $M$ be a subset of $X$ and $m_0 \in M$. Then, $M$ is a $\mathcal{C}^p$-submanifold at $m_0$ if and only if there exist a closed subspace $Z$, two open neighborhoods $U \in \mathcal{N}_X(m_0)$ and $V \in \mathcal{N}_Z(0)$, and a mapping $\theta : V \to Z^\perp$ of class $\mathcal{C}^p$ such that $\theta(0) = 0$, $D\theta(0) = 0$ and*

$$M \cap U = (L^{-1}(\mathrm{gph}\, \theta) + m_0) \cap U,$$

*where $L : X \to Z \times Z^\perp$ is the canonic isomorphism given by $L(x) = (\pi_Z(x), \pi_{Z^\perp}(x))$. In such a case, $Z = T_{m_0}M$.*

*Proof.* Let us prove first the sufficiency. By Example 4.1.3.(c), we have that $\operatorname{gph}\theta$ is a $\mathcal{C}^p$-submanifold of $Z \times Z^\perp$ with model space $Z$. Noting that the mapping $\varphi : Z \times Z^\perp \to X$ given by $\varphi(\cdot) = L^{-1}(\cdot) + m_0$, is a $\mathcal{C}^\infty$-diffeomorphism, we have, using Example 4.1.3.(d), that $L^{-1}(\operatorname{gph}\theta) + m_0$ is a $\mathcal{C}^p$-submanifold. Finally, noting that $m_0 \in (L^{-1}(\operatorname{gph}\theta) + m_0) \cap U$, the conclusion follows. Furthermore, we can write

$$T_{m_0}M = D\varphi(0)T_{(0,0)}(\operatorname{gph}\theta) = L^{-1}(Z \times \{0\}) = Z,$$

And so, the second part of the proposition also holds.

Now, let us prove the necessity. Since $M$ is a $\mathcal{C}^p$-submanifold at $m_0$, there exist an open neighborhood $W \in \mathcal{N}_X(m_0)$, a closed subspace $Z$ of $X$ and a $\mathcal{C}^p$-diffeomorphism $\varphi : W \to \varphi(W) \subseteq X$ such that $\varphi(m_0) = 0$ and that $\varphi(W \cap M) = \varphi(W) \cap Z$. Replacing $\varphi$ by $D\varphi^{-1}(0) \circ \varphi$ if necessary and using equation (4.1), we may and do assume that $Z = T_{m_0}M$.

Consider now the function $\phi : \varphi(W) \cap Z \to Z$ given by $\phi(z) = \pi_Z(\varphi^{-1}(z) - m_0)$. Since $D\phi(0) = D\varphi^{-1}(0)\big|_Z$ is an isomorphism from $Z$ to $Z$, we get by the Local Inverse Function Theorem, that there exists an open neighborhood $O \in \mathcal{N}_Z(0)$ such that $\phi : O \to \phi(O)$ is a $\mathcal{C}^p$-diffeomorphism.

Choose $\delta > 0$ small enough such that $B_X(0,\delta) \subseteq W$ and $Z \cap B_X(0,\delta) \subseteq O$ and fix $U := \varphi^{-1}(B_X(0,\delta))$ and $V := \phi(Z \cap B_X(0,\delta))$. We have that

$$\varphi(M \cap U) = Z \cap \varphi(U) = Z \cap B_X(0,\delta) = \phi^{-1}(V). \tag{4.2}$$

Define now $\theta : V \to Z^\perp$ as $\theta := \pi_{Z^\perp}(\varphi^{-1} \circ \phi^{-1}(\cdot) - m_0)\big|_V$. Clearly, $\theta(0) = 0$ and

$$D\theta(0) = \pi_{Z^\perp} \circ D\varphi^{-1}(0) \circ D\phi^{-1}(0) = 0,$$

since $D\varphi^{-1}(0) \circ D\phi^{-1}(0)Z = Z$. Also, for every $m \in U \cap M$ we can write

$$L(m - m_0) = (\pi_Z(\varphi^{-1} \circ \phi^{-1}(v) - m_0), \pi_{Z^\perp}(\varphi^{-1} \circ \phi^{-1}(v) - m_0)) = (v, \theta(v)),$$

where $v := \phi \circ \varphi(m)$ (which is in $V$, by equation (4.2)). Therefore, $U \cap M \subseteq (L^{-1}(\operatorname{gph}\theta) + m_0) \cap U$. For the other inclusion, take $v \in V$ such that $L^{-1}(v, \theta(v)) + m_0 \in U$. Again, by equation 4.2, we get that there exists $m \in M \cap U$ such that $v := \phi \circ \varphi(m)$. Therefore,

$$L^{-1}(v, \theta(v)) = \phi \circ \phi^{-1}(v) + \theta(v) = \varphi^{-1} \circ \phi^{-1}(v) - m_0 = m - m_0.$$

Thus, $L^{-1}(v, \theta(v)) + m_0 \in M \cap U$, finishing the proof of the equivalence. The proof is then complete. □

The next proposition provides conditions assuming that the level sets of a function of class $\mathcal{C}^p$ are $\mathcal{C}^p$-submanifolds. This proposition will be very useful when we will study the smoothness of the distance function to a set.

**Proposition 4.1.5** *Let $U_0$ be an open set of $X$, a point $\bar{x} \in U_0$, a Banach space $Y$ and a mapping $g : U_0 \to Y$ of class $\mathcal{C}^p$. If $Dg(\bar{u})$ is surjective, then for $\bar{y} := g(\bar{u})$, the set $M = \{x \in U_0 \ : \ g(x) = \bar{y}\}$ is a $\mathcal{C}^p$-submanifold at $\bar{u}$, with $T_{\bar{u}}M = \mathrm{Ker}(Dg(\bar{u}))$.*

*Proof.* Without loss of generality, we may assume that $\bar{y} = 0$. Let us denote $X_2 = \mathrm{Ker}(Dg(\bar{x}))$, $X_1 = X_2^{\perp}$ and by $\pi_{X_1}$ and $\pi_{X_2}$ the associated parallel projections. By the Local Submersion theorem (see [1, Theorem 2.5.12]), there exists an open neighborhood $U \subseteq U_0$ of $\bar{x}$ in $X$, an open neighborhood $V$ of $(g(\bar{x}), \pi_{X_2}(\bar{x}))$ in $Y \times X_2$, and a $\mathcal{C}^p$-diffeomorphism $\psi : V \to U$ such that

$$g \circ \psi(v_1, v_2) = v_1, \ \forall (v_1, v_2) \in V.$$

Since the bounded linear operator $A_0 : X_1 \to Y$ given by $A_0(x_1) = Dg(\bar{x})x_1$ is bijective, hence an isomorphism between $X_1$ and $Y$ according to the Closed Graph Theorem, the mapping $j : X_1 \oplus X_2 \to Y \times X_2$ given by $j(x_1 + x_2) = (A_0(x_1), x_2)$ is also an isomorphism. Define $j_V : j^{-1}(V) \to V$ as the bijective restriction of $j$ to $j^{-1}(V)$, and consider the $\mathcal{C}^p$-diffeomorphism $\varphi := j_V^{-1} \circ \psi^{-1}$ from $U$ onto $j^{-1}(V)$. Then, we can write

$$\begin{aligned}
x \in \varphi(U \cap M) &\Leftrightarrow \psi \circ j_V(x) \in U \text{ and } g \circ \psi \circ j_V(x) = 0 \\
&\Leftrightarrow \psi \circ j_V(x) \in U \text{ and } g \circ \psi(A_0(\pi_{X_1}(x)), \pi_{X_2}(x)) = 0 \\
&\Leftrightarrow \psi \circ j_V(x) \in U \text{ and } A_0(\pi_{X_1}(x)) = 0 \\
&\Leftrightarrow \psi \circ j_V(x) \in U \text{ and } \pi_{X_1}(x) = 0 \\
&\Leftrightarrow x \in \varphi(U) \cap X_2,
\end{aligned}$$

which means that $M$ is a $\mathcal{C}^p$-submanifold at $\bar{x}$ with model space $X_2$. Finally, we know that $T_{\bar{x}}M = D\varphi^{-1}(\varphi(\bar{x}))X_2$. Noting that

$$Dg(\bar{x}) \circ D\varphi^{-1}(\varphi(\bar{x})) = D(g \circ \varphi^{-1})(\varphi(x)) = A_0 \circ \pi_{X_1} = Dg(\bar{x}),$$

we get that $T_{\bar{x}}M \subseteq \mathrm{Ker}\, Dg(\bar{x})$. On the other hand, we know that $Dg(\bar{x}) = \pi_Y \circ D\psi^{-1}(\bar{x})$ and so, if $x \in \mathrm{Ker}\, Dg(\bar{x})$, then $D\psi^{-1}(\bar{x})x \in X_2$. We conclude that

$$D\varphi(\bar{x}) \,\mathrm{Ker}\, Dg(\bar{x}) = J_V^{-1} \circ D\psi^{-1}(\bar{x}) \,\mathrm{Ker}\, Dg(\bar{x}) \subseteq X_2,$$

which means that $T_{\bar{x}}M = D\varphi(\bar{x})^{-1}X_2 \supseteq \mathrm{Ker}\, Dg(\bar{x})$, finishing the proof. □

To close this section, we will introduce the notion of differentiable mappings between submanifolds. Even though there are several ways to define the differentiability of such mappings, we will follow the one introduced in [4, Ch. 9 - Section 3].

**Definition 4.1.6** (Differentiable mappings) *Let $Y$ be another Hilbert space, $M$ be a $\mathcal{C}^p$-submanifold of $X$, $N$ be a $\mathcal{C}^p$-submanifold of $Y$ and $f : M \to N$ be a continuous mapping (with respect to the relative topologies on $M$ and $N$). We say that $f$ is differentiable at a point $m \in M$ (resp. of class $\mathcal{C}^k$ near $m$, with $1 \leq k \leq p$) if there exists $U \in \mathcal{N}_X(m)$ and a mapping $\hat{f} : U \to Y$ such that* *(a) $\hat{f}$ is differentiable at $m$ (resp. of class $\mathcal{C}^k$ near $m$); and*

*(b) $\hat{f}\big|_{U\cap M} = f\big|_{U\cap M}$.*

*If $f$ is differentiable at each point of $M$ (resp. of class $\mathcal{C}^k$ near each point of $M$), we simply say that $f$ is differentiable (resp. of class $\mathcal{C}^k$).*

When $f : M \to N$ is a differentiable mapping between submanifolds, the derivative of $f$ at $m \in M$, called the *Tangent (linear) mapping* of $f$ at $m$, is given by

$$T_m f := D\hat{f}(m)\big|_{T_m M}.$$

This definition doesn't depend on the extension $\hat{f}$ chosen and also we have that $T_m f(T_m M)$ is included in $T_{f(m)}N$. Indeed, let $\hat{f}$ be such an extension and consider $h \in T_m M$. By Definition 4.1.2, there exists a $\mathcal{C}^1$-curve $\gamma : ]-1,1[\to M$ with $\gamma(0) = m$ and $\gamma'(0) = h$. Now, we have that $f \circ \gamma = \hat{f} \circ \gamma$ and it is a $\mathcal{C}^1$-curve in $N$ with $f \circ \gamma(0) = f(m)$. Then, by chain rule, we have that

$$(f \circ \gamma)'(0) = D\hat{f}(m)\gamma'(0) = D\hat{f}(m)h.$$

Then, the restriction of $D\hat{f}(m)$ to $T_m M$ is unique, not depending on the extension chosen, and $T_m f : T_m M \to T_{f(m)}N$ is the unique continuous linear mapping such that $T_m f \circ \gamma'(0) = (f \circ \gamma)'(0)$ for every $C^1$-curve $\gamma : ]-1,1[\to M$ with $\gamma(0) = m$.

We will state two results concerning differentiable mappings that can be found in [4, Ch. 9 - Section 3], namely, the chain rule and the Local Inverse Theorem:

**Proposition 4.1.7** (Chain rule) *Let $M_1, M_2$ and $M_3$ be three $\mathcal{C}^p$-submanifolds of three Hilbert spaces $X_1, X_2$ and $X_3$, respectively. Let also $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be two continuous mappings. If $f$ is differentiable at $m_1 \in M_1$ (resp. of class $\mathcal{C}^k$ near $m_1$, with $1 \le k \le p$) and $g$ is differentiable at $f(m_1) \in M_2$ (resp. of class $\mathcal{C}^k$ near $f(m_1)$), then $g \circ f$ is differentiable at $m_1$ (resp. of class $\mathcal{C}^k$ near $m_1$) with*

$$T_{m_1}(g \circ f) = T_{f(m_1)}g \circ T_{m_1} f.$$

*Proof.* It is enough to consider, according to Definition 4.1.6, two extensions $\hat{f}$ and $\hat{g}$ of $f$ and $g$, respectively, and apply the classical chain rule between Banach spaces. □

**Theorem 4.1.8** (Local Inverse Theorem) *Let $M$ and $N$ be two $\mathcal{C}^p$-submanifolds of two Hilbert spaces $X$ and $Y$ respectively, and consider a mapping $f : M \to N$ which is of class $\mathcal{C}^k$ (with $1 \le k \le p$) near $m \in M$. If $T_m f$ is bijective, then $f$ is a local $\mathcal{C}^k$-diffeomorphism between $M$ and $N$ near $m$, that is, there exists a neighborhood $U \in \mathcal{N}_X(m)$ and a neighborhood $V \in \mathcal{N}_Y(f(m))$ such that the function $f\big|_U : U \to V$ is bijective and $f\big|_U$, $\left(f\big|_U\right)^{-1}$ are both of class $\mathcal{C}^k$.*

## 4.2 Proximal calculus and Prox-regular sets

In Hilbert spaces, the geometry of many sets can be described using *proximal normal vectors*. These normal vectors are defined starting from the notion of *nearest points*, which are known to always exists for convex closed sets. In this section we will give a quick overview of the theory of Proximal Calculus and its relationship with the well-known Clarke's theory for nonsmooth calculus (namely, his Tangent cone, Normal cone and subdifferential). We will also introduce the notion of Prox-regular sets and we will describe some of their properties. Most of the results of this sections can be found in the books [3], [10] and [23].

### 4.2.1 Proximal Normal Cone and Proximal Subdifferential

First, let us recall the notions of nearest points and *metric projections*. In the following, for a nonempty subset $S$ of $X$, we will denote by $d_S(\cdot)$ (or $d(\cdot; S)$) the distance function to $S$, that is, for $x \in X$

$$d_S(x) := \inf\{\|x - s\| \ : \ s \in S\}.$$

**Definition 4.2.1** (Metric Projection) *Let $S$ be a nonempty subset of $X$ and $x \in X$ be a fixed point. We say that a point $s \in S$ is a* nearest point *or a* projection *of $x$ onto $S$ if*

$$\|x - s\| = d_S(x).$$

*The set of all projections of $x$ onto $S$ is denoted by* $\mathrm{Proj}_S(x)$ *or* $\mathrm{Proj}(x; S)$. *If* $\mathrm{Proj}_S(x)$ *is a singleton, then the unique nearest point of $x$ onto $S$ is called the* metric projection *of $x$ onto $S$ and it is denoted by $P_S(x)$.*

It is a classic result that if $S$ is closed and convex, then for every $x \in X$, the metric projection of $x$ onto $S$ exists (and is unique), and so the mapping $P_S : X \to S$ is well-defined. If we drop the hypothesis of convexity, the set $\mathrm{Proj}_S(x)$ can have many points or even be empty.

Geometrically speaking, we can interpret the set $\mathrm{Proj}_S(x)$ as the collection of all points of the set $S$ where it is tangent to the ball centered in $x$ with radius $d_S(x)$, and so

$$s \in \mathrm{Proj}_S(x) \iff s \in S \cap B_X[x, \|x - s\|] \text{ and } S \cap B_X(x, \|x - s\|) = \emptyset. \tag{4.3}$$

**Proposition 4.2.2** (See [10, Proposition 1.3]) *Let $S$ be a nonempty subset of $X$, and let $x \in X$, $s \in S$. The following assertions are equivalent:*

*(i)* $s \in \mathrm{Proj}_S(x)$*;*

*(ii) For all* $t \in [0, 1]$, $s \in \mathrm{Proj}_S(s + t(x - s))$*;*

*(iii)* *For all* $t \in [0,1]$, $d_S(s+t(x-s)) = t\|x-s\|$;

*(iv)* *For all* $s' \in S$, $\langle x-s, s'-s\rangle \leq \frac{1}{2}\|s'-s\|^2$.

*In such a case, we have that for every* $t \in ]0,1[$, $\mathrm{Proj}_S(s+t(x-s)) = \{s\}$.

**Definition 4.2.3** (Proximal normal cone) *Let* $S$ *be a nonempty subset of* $X$ *and* $\bar{s} \in S$. *We say that* $\zeta \in X$ *is a* proximal normal vector *to* $S$ *at* $\bar{s}$ *if there exists* $t > 0$ *such that*

$$\bar{s} \in \mathrm{Proj}_S(\bar{s} + t\zeta).$$

*The set of all proximal normal vectors to* $S$ *at* $\bar{s}$ *is called the* Proximal Normal Cone *of* $S$ *at* $\bar{s}$ *and is denoted by* $N^P(S;\bar{s})$.

By convention, $N^P(S;x) = \emptyset$ whenever $x \notin S$, and so $N^P(S;\cdot)$ defines a multifunction from $X$ into $X$. Also, it is clear that if $s \in \operatorname{int} S$, then $N^P(S;\bar{s}) = \{0\}$. Intuitively, $N^P(S;\bar{s})$ is the set of all the directions of *perpendicular departure* from $S$ at the point $\bar{s}$. Also, we have the equality

$$N^P(S;\bar{s}) = \{\zeta \in X \ : \ \exists t > 0, \ d_S(\bar{s}+t\zeta) = t\|\zeta\|\}.$$

Here we will introduce the notions of *open normal ray* and $\lambda$*-truncated open normal ray.* They will be used mainly in Chapter 5 in order to state the generalizations of Holmes' Theorem and its converse.

**Definition 4.2.4** (Open Normal Ray) *Let* $S$ *be a closed set of* $X$, $x \in \operatorname{bd} S$ *and* $\lambda > 0$. *If the proximal normal cone of* $S$ *at* $x$ *is of the form*

$$N^P(S;x) = \{t\nu \ : \ t \in \mathbb{R}_+\},$$

*for some unit vector* $\nu \in \mathbb{S}_X$, *we define the* open normal ray *of* $S$ *at* $x$ *and the* $\lambda$-truncated open normal ray *of* $S$ *at* $x$ *as the sets*

$$\mathrm{Ray}_x(S) := \{x + t\nu \ : \ t \in ]0,+\infty[\,\} \tag{4.4}$$

$$\mathrm{Ray}_{x,\lambda}(S) := \{x + t\nu \ : \ t \in ]0,\lambda[\,\} \tag{4.5}$$

*respectively.*

Motivated by characterization $(iv)$ of $\mathrm{Proj}_S(\cdot)$, it is reasonable to expect similar variational inequalities to describe the proximal normal cone. We also expect the proximal normality to be a local property. These conditions are contained in the following proposition.

**Proposition 4.2.5** (See [10, Proposition 1.5]) *Let* $S$ *be a nonempty subset of* $X$, $\delta > 0$ *and let* $s \in S$ *and* $\zeta \in X$. *The following assertions are equivalent:*

*(i)* $\zeta \in N^P(S;s)$.

*(ii) There exists $\sigma \geq 0$ depending on $\zeta$ and $s$ such that*

$$\langle \zeta, s' - s \rangle \leq \sigma \|s' - s\|^2, \ \forall s' \in S.$$

*(iii) There exists $\sigma \geq 0$ depending on $\zeta$, $s$ and $\delta$ such that*

$$\langle \zeta, s' - s \rangle \leq \sigma \|s' - s\|^2, \ \forall s' \in B_X(s, \delta) \cap S.$$

*In particular, for any $\delta > 0$, $N^P(S; s) = N^P(B_X(s, \delta) \cap S, s)$.*

The above characterization happens to be very useful when we want to obtain properties of the Proximal normal cone. From it, we can derive the following corollary.

**Corollary 4.2.6** *Let $S$ be a nonempty subset of $X$ and let $x \in S$. We have that $N^P(S; x)$ is convex.*

Unfortunately, even though we have convexity, the proximal normal cone is not necessarily neither open nor closed. Proposition 4.2.5 also entails the following property of invariance under isomorphisms, which will be very useful in the development of Chapter 5.

**Proposition 4.2.7** *Let $Y$ be a Hilbert space, $S'$ be a nonempty subset of $Y$ and $A : X \to Y$ be a bijective continuous linear mapping. For every $x \in A^{-1}(S')$ we have that*

$$N^P(A^{-1}(S'); x) = A^* N^P(S'; A(x)) = \{A^*\zeta \ : \ \zeta \in N^P(S'; A(x))\}. \tag{4.6}$$

*Proof.* To simplify notation, set $S := A^{-1}(S')$. Let $\xi \in N^P(S; x)$. We have that there exists $\sigma > 0$ such that

$$\langle \xi, x' - x \rangle \leq \sigma \|x' - x\|^2, \ \forall x' \in S.$$

Then, recalling that $(A^*)^{-1} = (A^{-1})^*$ for all $y \in S'$ we can write

$$\begin{aligned} \langle (A^*)^{-1}\xi, y - A(x) \rangle &= \langle (A^{-1})^*\xi, y - A(x) \rangle \\ &= \langle \xi, A^{-1}(y) - x \rangle \\ &\leq \sigma \|A^{-1}(y) - x\|^2 \leq \sigma \|A^{-1}\|^2 \|y - A(x)\|^2. \end{aligned}$$

Then, by Proposition 4.2.5, we get that $(A^*)^{-1}\xi \in N^P(S'; A(x))$, concluding that $N^P(S; x)$ is contained in $A^* N^P(S'; A(x))$. The reverse inclusion follows by symmetry, replacing $A$ by $A^{-1}$. □

As usual, these geometric objects defined for sets can be extended to functions. This motivates the next definition.

**Definition 4.2.8** (Proximal subdifferential) *Let $f : X \to \mathbb{R}_\infty$ be a proper function, and let $x \in \operatorname{dom} f$. A vector $\zeta \in X$ is called a* proximal subgradient *of $f$ at $x$ if*

$$(\zeta, -1) \in N^P(\operatorname{epi} f; (x, f(x))).$$

*The set of all these proximal subgradients is called the* proximal subdifferential *of $f$ at $x$, denoted by $\partial_P f(x)$.*

By convention, we will set $\partial_P f(x) = \emptyset$, whenever $x \notin \operatorname{dom} f$, and so $\partial_P f(\cdot)$ defines a multifunction from $X$ into $X$. Also, it is not hard to see that, for a nonempty set $S \subseteq X$

$$\partial_P I_S(x) = N^P(S; x), \ \forall x \in X,$$

where $I_S$ is the indicator function of $S$ (see Introduction of Part II). We also can prove the Fermat's rule for the proximal subdifferential, namely, if a proper function $f : X \to \mathbb{R}_\infty$ attains a local minimum at $x \in \operatorname{dom} f$, then $0 \in \partial_P f(x)$. Indeed, since $x$ is a local minimum of $f$, there exists a neighborhood $U \in \mathcal{N}_X(x)$ such that,

$$\forall x' \in U, \ f(x) \leq f(x').$$

Therefore, for every $(x', r') \in \operatorname{epi} f \cap (U \times \mathbb{R})$, we have that

$$\langle (0, -1), (x', r') - (x, f(x)) \rangle = f(x) - r' \leq f(x) - f(x') \leq 0 = 0 \|(x', r') - (x, f(x))\|^2.$$

Therefore, by Proposition 4.2.5, we get that $(0, -1) \in N^P(\operatorname{epi} f, (x, f(x)))$, proving our claim. To avoid pathological situations, we will restrict our study to nonempty closed sets and so, in the functional context, to lower semicontinuous functions (since they are characterized by the closedness of their epigraphs).

**Proposition 4.2.9** (See See [10, Ch. 1 - Theorem 2.5]) *Let $f : X \to \mathbb{R}_\infty$ be a proper lower semicontinuous function and let $x \in \operatorname{dom} f$. A vector $\zeta \in X$ belongs to $\partial_P f(x)$ if and only if there exists two positive numbers $\sigma$ and $\eta$ such that*

$$f(y) \geq f(x) + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2, \ \forall y \in B_X(x, \eta) \tag{4.7}$$

Observe that Proposition 4.2.9 entails that whenever the function $f$ is $\gamma$-Lipschitz continuous near $x$ (with $\gamma > 0$), each element $\zeta \in \partial_P f(x)$ has norm $\|\zeta\| \leq \gamma$. Indeed, from equation (4.7), for each $\delta \in ]0, \eta[$ we can write

$$\gamma \geq \frac{|f(y) - f(x)|}{\|y - x\|} \geq \sup_{y \in B_X(x,\delta) \setminus \{x\}} \left\langle \zeta, \frac{y - x}{\|y - x\|} \right\rangle - \sigma \|y - x\| \geq \|\zeta\| - \sigma\delta,$$

which, by arbitrariness of $\delta$, proves our claim.

Another implication of this variational characterization, is the behavior of $\partial_P f$ when $f$ is differentiable in some sense, which we describe in the following corollary.

**Corollary 4.2.10** (See [10, Corollary 2.6]) *Let $f : X \to \mathbb{R}_\infty$ be a proper lower semicontinuous function, $U$ be an open set of $X$ and let $x \in U$. We have that*

*(a) If $f$ is G-differentiable at $x$, then $\partial_P f(x) \subseteq \{\nabla_G f(x)\}$.*

(b) *If $f$ is of class $\mathcal{C}^2$ at $x$, then $\partial_P f(x) = \{\nabla f(x)\}$.*

If $f$ is of class $\mathcal{C}^1$ at $x$, we still could get that $\partial_P f(x)$ is empty, as we can see in [10, Ch. 1 - Exercise 1.7].

We end this section establishing a useful relation between the Proximal subdifferential of the distance function and the enlargements of sets. For $\lambda > 0$ and a nonempty subset $S$ of $X$ let us define the $\lambda$-*enlargement* of $S$ as the set

$$S_\lambda := \{x \in X \;:\; d_S(x) \leq \lambda\} \quad (= d_S^{-1}([0,\lambda])). \tag{4.8}$$

The proof of the next proposition is mainly contained in [5].

**Proposition 4.2.11** *Let $S$ a nonempty subset of $X$. The following hold:*

(a) *For $x \in \operatorname{cl} S$ we have that*

$$\partial_P d_S(x) = N^P(\operatorname{cl} S; x) \cap \mathbb{B}_X.$$

(b) *For $x \notin \operatorname{cl} S$, fixing $\lambda := d_S(x)$, we have that*

$$\partial_P d_S(x) = N^P(S_\lambda; x) \cap \mathbb{S}_X.$$

### 4.2.2 Clarke's Normal Cone and Normal regularity

The Proximal Normal Cone is not the unique way to extend the normal cone to the nonconvex setting. In the literature, we can find several notions, as for example the *Fréchet Normal Cone* or the *Limiting Normal Cone* (see, e.g. [21], [23]). In this section, we will focus on the Clarke Normal Cone, which is defined from the *Clarke Tangent Cone*. These two objects were introduced by Clarke after defining a new subdifferential to work with outside the convex setting. Instead of the historical development, we will follow a much modern approach based on the lecture's notes of L. Thibault (see [30]).

Let us recall first that the notion of Peano-Painlevé-Kuratowski limits of sets. Let $(T,\tau)$ and $(S,\sigma)$ be two Hausdorff topological spaces, $M : T \rightrightarrows S$ be a multifunction, $t_0 \in T$ and $T_0 \subseteq T$ such that $t_0 \in \operatorname{cl}(T_0)$. We define the *Peano-Painlevé-Kuratowski inferior and superior limits* of $M$ at $t_0$ relative to $T_0$ as

$$\operatorname*{Liminf}_{T_0 \ni t \to t_0} M(t) := \{s \in S \;:\; \forall W \in \mathcal{N}_S(s),\ \exists V \in \mathcal{N}_T(t_0),\ \forall t \in V \cap T_0,\ M(t) \cap W \neq \emptyset\}.$$
$$\operatorname*{Limsup}_{T_0 \ni t \to t_0} M(t) := \{s \in S \;:\; \forall W \in \mathcal{N}_S(s),\ \forall V \in \mathcal{N}_T(t_0),\ \exists t \in V \cap T_0,\ M(t) \cap W \neq \emptyset\}.$$

In general, $\operatorname*{Liminf}_{T_0 \ni t \to t_0} M(t) \subset \operatorname*{Limsup}_{T_0 \ni t \to t_0} M(t)$. If the other inclusion also holds, the common set is called the *Peano-Painlevé-Kuratowski limit* of $M$ at $t_0$ relative to $T_0$, and it is denoted by $\operatorname*{Lim}_{T_0 \ni t \to t_0} M(t)$. We refer the reader to [23, Section 1.3] for further information on these notions.

**Definition 4.2.12** (Clarke Tangent and Normal Cones) *Let $S$ be a nonempty set of $X$ and $x_0$ be a point in $S$. We define the* Clarke Tangent Cone *of $S$ at $x_0$ as the set*

$$T^C(S;x_0) := \operatorname*{Liminf}_{S \ni u \to x_0;\ t \downarrow 0} \tfrac{1}{t}(S-u),$$

*where, in the definition of inferior limit, $S \times \mathbb{R}_+$, $(x_0,0) \in S \times \mathbb{R}_+$, $S\times\ ]0,+\infty[$ and $(u,t) \rightrightarrows \frac{1}{t}(S-u)$ play the role of the topological space $T$, the limit point $t_0$, the set $T_0$ and the multifunction $M$, respectively.*

*We also define the* Clarke Normal Cone *of $S$ at $x_0$, denoted by $N^C(S;x_0)$, as the negative polar set of the Clarke Tangent Cone, that is,*

$$N^C(S;x_0) := \left[T^C(S;x_0)\right]^o = \{\zeta \in X \ :\ \langle \zeta, h\rangle \leq 0,\ \forall h \in T^C(S;x_0)\}.$$

The Clarke tangent and normal cones enjoy several nice properties that make them a powerful tool in nonconvex optimization and variational analysis. We give a quick overview of some of those properties, concentrating us in those which we will use.

The next proposition can be easily derived from Definition 4.2.12.

**Proposition 4.2.13** (Sequential characterization of Clarke tangent cone) *Let $S$ be a nonempty set of $X$ and let $x_0, h \in X$. The following assertions are equivalent:*

*(i)* $h \in T^C(S;x_0)$.

*(ii) For any sequence $(x_n) \subset S$ converging to $x_0$ and any sequence $(t_n) \subseteq\ ]0,+\infty[$ converging to $0$, there exists a sequence $(h_n) \subseteq X$ converging to $h$ such that*

$$x_n + t_n h_n \in S, \text{ for all } n \in \mathbb{N} \text{ (or for all } n \text{ large enough).}$$

*(iii) For any sequence $(x_n) \subset S$ converging to $x_0$ and any sequence $(t_n) \subseteq\ ]0,+\infty[$ converging to $0$, there exist an increasing function $\psi : \mathbb{N} \to \mathbb{N}$ and a sequence $(h_n) \subseteq X$ converging to $h$ such that*

$$x_{\psi(n)} + t_{\psi(n)} h_n \in S, \text{ for all } n \in \mathbb{N}.$$

Next proposition surveys some useful properties of the Clarke tangent and normal cones. It is difficult to find it in its present form, but all results can be derived from what is contained in [3, Ch. 4], [10, Ch. 2] and [23, Ch. 5].

**Proposition 4.2.14** *Let $S$ be a nonempty set and let $x_0 \in S$. We have that*

(a) $T^C(S;x_0)$ *and* $N^C(S;x_0)$ *are convex closed cones and so*

$$T^C(S;x) = \left[N^C(S;x)\right]^o.$$

(b) *The Clarke tangent and normal cones of* $S$ *at* $x_0$ *are* local, *that is, for any neighborhood* $U \in \mathcal{N}_X(x_0)$ *we have that*

$$T^C(S\cap U;x_0) = T^C(S;x_0) \qquad \textit{and} \qquad N^C(S\cap U;x_0) = N^C(S;x_0).$$

*Furthermore, for any other set* $S'$ *such that* $S\cap U \subseteq S'\cap U \subseteq \mathrm{cl}(S)\cap U$, *we have that*

$$T^C(S';x_0) = T^C(S;x_0) \qquad \textit{and} \qquad N^C(S';x_0) = N^C(S;x_0).$$

(c) *If* $Y$ *is another Hilbert space,* $U \subseteq X$ *is an open neighborhood of* $x_0$, $\varphi : U \to \varphi(U) \subseteq Y$ *is a* $\mathcal{C}^1$*-diffeomorphism, and* $S' \subseteq \varphi(U)$ *is such that* $U \cap S = \varphi^{-1}(S')$, *then*

$$T^C(S;x_0) = T^C(\varphi^{-1}(S');x_0) = D\varphi(x_0)^{-1}\left(T^C(S',\varphi(x_0))\right). \tag{4.9}$$

$$N^C(S;x_0) = N^C(\varphi^{-1}(S');x_0) = D\varphi(x_0)^{*}\left(N^C(S',\varphi(x_0))\right). \tag{4.10}$$

(d) *If in addition* $x_0 \in \mathrm{bd}\, S$, *the equality*

$$T^C(\mathrm{bd}\, S;x_0) = T^C(S;x_0) \cap T^C(S^c\cup\{x_0\};x_0)$$

*always hods.*

As we did with the Proximal normal cone, the Clarke normal cone induces a subdifferential notion, which is called the *Clarke subdifferential* or the *generalized subgradient* and it was introduced by F. H. Clarke in 1975 in order to generalize the convex subdifferential to Lipschitz functions (see [8], [9]).

**Definition 4.2.15** (Clarke subdifferential) *Let* $f : X \to \mathbb{R}_\infty$ *be a proper function and let* $x \in \mathrm{dom}\, f$. *The* Clarke subdifferential *of* $f$ *at* $x$ *is defined as*

$$\partial_C f(x) := \left\{\zeta \in X \;:\; (\zeta,-1) \in N^C(\mathrm{epi}\, f;(x,f(x)))\right\}.$$

By convention, we will set $\partial_C f(x) = \emptyset$ whenever $x \notin \mathrm{dom}\, f$. Thus, $\partial f(\cdot)$ defines a multifunction from $X$ onto $X$. We will know describe some properties of Clarke subdifferential, restricting us only to those we will use.

**Proposition 4.2.16** *Let* $f : X \to \mathbb{R}_\infty$ *be a proper function and let* $x \in \mathrm{dom}\, f$. *We have that*

(a) $\partial_C f(x)$ *is a convex closed set (possibly empty).*

*(b) If $x$ is a local minimum of $f$, then $0 \in \partial_C f(x)$.*

*(c) If $f$ is F-differentiable at $x$, then $\nabla f(x) \in \partial_C f(x)$. If in addition $f$ is of class $\mathcal{C}^1$ at $x$, then $\partial_C f(x) = \{\nabla f(x)\}$.*

*(d) If $f = I_S$ for some subset $S$ of $X$, then*

$$\partial_C I_S(x) = N^C(S; x).$$

When the function $f$ is Lipschitz-continuous near $x$, the Clarke subdifferential has further properties and it is strongly related with the so called *Clarke directional derivative.*

**Definition 4.2.17** (Clarke directional derivative) *Let $f : X \to \mathbb{R}_\infty$ a proper function and let $x \in \operatorname{dom} f$. If $f$ is Lipschitz-continuous near $x$, we define the* Clarke directional derivative *of $f$ at $x$ as the function $f^o(x; \cdot) : X \to \mathbb{R}_\infty$ given by*

$$f^o(x; v) := \limsup_{x' \to x;\ t \downarrow 0} \frac{f(x' + tv) - f(x')}{t}.$$

**Proposition 4.2.18** *Let $f : X \to \mathbb{R}_\infty$ be a proper function and let $x \in \operatorname{dom} f$. If $f$ is Lipschitz-continuous near $x$ with Lipschitz-constant $\gamma > 0$, then*

*(a) $f^o(x; \cdot)$ is well-defined, sublinear and $\gamma$-Lipschitz on $X$. Furthermore,*

$$T^C(\operatorname{epi} f; (x, f(x))) = \operatorname{epi} f^o(x; \cdot).$$

*(b) $\partial_C f(x)$ is nonempty and $w^*$-compact and for all $\xi \in \partial_C f(x)$, $\|\xi\| \le \gamma$.*

*(c) For every $\xi \in X$, $\xi \in \partial_C f(x)$ if and only if $\langle \xi, v \rangle \le f^o(x; v)$ for all $v \in X$. Furthermore,*

$$f^o(x; v) = \max\{\langle \xi, v \rangle \ : \ \xi \in \partial_C f(x)\}.$$

Observe that the proof of the latter proposition entails that, when $f$ is Lipschitz-continuous near $x$, we can write the equality

$$N^C(\operatorname{epi} f; (x, f(x))) = \mathbb{R}_+ \left[\partial_C f(x) \times \{-1\}\right], \tag{4.11}$$

which in particular, ensures that there are no horizontal normal directions to the epigraph of $f$ at the point $(x, f(x))$, that is, whenever $(\xi, s) \in N^C(\operatorname{epi} f; (x, f(x))) \setminus \{(0, 0)\}$, then necessarily $s < 0$.

Before establishing the relationship between the Clarke normal cone and the Proximal normal cone, we will introduce a third concept of normal cone, which is related to the so called *Bouligand tangent cone* or *contingent cone.*

**Definition 4.2.19** (Bouligand tangent and normal cones) *Let $S$ be a nonempty subset of $X$ and let $x_0 \in S$. We define the* Bouligand tangent cone *of $S$ at $x_0$ as the set*

$$T^B(S; x_0) := \operatorname*{Limsup}_{t \searrow 0} \tfrac{1}{t}(S - x_0),$$

*where, in the definition of superior limit, $\mathbb{R}_+$, $0 \in \mathbb{R}_+$, $]0, +\infty[$ and $t \rightrightarrows \frac{1}{t}(S - x_0)$ play the role of the topological space $T$, the limit point $t_0$, the set $T_0$ and the multifunction $M$, respectively.*

*We also define the* Bouligand Normal Cone *of $S$ at $x_0$, denoted by $N^B(S; x_0)$, as the negative polar set of the Bouligand tangent cone.*

The Bouligand tangent cone of $S$ at $x_0$ can be equivalently defined as the set of all directions $d \in X$ for which there exist a sequence $(d_n) \subseteq X$ converging to $d$ and a sequence $(t_n) \subseteq ]0, +\infty[$ converging to 0 such that

$$x_0 + t_n d_n \in S, \ \forall n \in \mathbb{N}. \tag{4.12}$$

Some properties that are not hard to verify are contained in the following proposition, which we will leave without proof. We refer the reader to [3] and the references therein for more details.

**Proposition 4.2.20** *Let $S$ be a nonempty subset of $X$, and let $x_0 \in S$.*

*(a) $T^B(S; x_0)$ is a closed cone (but not necessarily convex);*

*(b) The Bouligand tangent and normal cones are local, namely, they satisfy the relations of Proposition 4.2.14.(b), established for the Clarke tangent and normal cones.*

*(c) The Bouligand tangent cone is* isotone, *that is, for every subset $S'$ of $X$ containing $S$, we get that*

$$T^B(S; x_0) \subseteq T^B(S', x_0).$$

An important difference between Bouligand and Clarke tangent cones, is that the Clarke tangent cone is neither *isotone* nor *antitone*, that is, the inclusion $S \subset S'$ with $x \in S$ does not imply either $T^C(S; x) \subset T^C(S'; x)$ (isotony) or $T^C(S; x) \supset T^C(S'; x)$ (antitony).

The relations in the general case between the Proximal normal cone, the Clarke cones and the Bouligand cones are summarized in the following proposition.

**Proposition 4.2.21** *Let $S$ be a nonempty subset of $X$ and let $x_0 \in S$. We have that*

*(a) $T^C(S; x_0) \subseteq T^B(S; x_0)$. Furthermore, if $S$ is convex, then*

$$T^C(S; x_0) = T^B(S; x_0) = \operatorname{cl}(\mathbb{R}_+(S - x_0)).$$

*(b)* $N^P(S;x_0) \subseteq N^B(S;x_0) \subseteq N^C(S;x_0)$. *Furthermore, if* $S$ *is convex, then*

$$N^P(S;x_0) = N^C(S;x_0) = \{x \in X \ : \ \langle x, u - x_0 \rangle \leq 0, \forall u \in S\}.$$

From Definitions 4.2.8 and 4.2.15, we can state the same inclusion of part (b) of the latter proposition for the Proximal and Clarke subdifferentials, namely, for a proper function $f : X \to \mathbb{R}_\infty$,

$$\partial_P f(x) \subseteq \partial_C f(x), \forall x \in X.$$

Likewise, if $f$ is convex, then all subdifferentials coincide, since in such a case we can write

$$\partial f(x) = \{\zeta \in X \ : \ f(x) + \langle \zeta, y - x \rangle \leq r, \ \forall (y,r) \in \operatorname{epi} f\}.$$

This motivates the following definition:

**Definition 4.2.22** *A nonempty subset* $S$ *of* $X$ *is said to be*

1. Tangentially regular *at* $x \in S$*, if* $T^C(S;x) = T^B(S;x)$.
2. Normally regular *at* $x \in S$ *if* $N^P(S;x) = N^C(S;x)$.

*Analogously, a proper function* $f : X \to \mathbb{R}_\infty$ *is said to be Tangentially regular (resp. Normally regular) at* $x \in \operatorname{dom} f$ *if its epigraph is Tangentially regular (resp. Normally regular) at* $(x, f(x))$.

Evidently, every convex set is both tangentially regular and normally regular at each of its points. Nevertheless, there are more regular sets (or functions) than the convex ones. Also it is worth pointing out that none of those two regularities implies the other one.

**Proposition 4.2.23** *Let* $f : X \to \mathbb{R}_\infty$ *be a proper function and let* $x \in \operatorname{dom} f$*. Assume that* $f$ *is Lipschitz-continuous near* $x$*. Then,* $f$ *is tangentially regular at* $x$ *if and only if the (classical) directional derivative* $f'(x;\cdot)$ *exists and it coincides with the Clarke directional derivative* $f^o(x;\cdot)$.

*In particular, if* $f$ *is of class* $\mathcal{C}^1$ *at* $x$*, then* $f$ *is tangentially regular at* $x$*. If in addition* $f$ *is of class* $\mathcal{C}^2$ *at* $x$*, then it is also normally regular at* $x$.

## 4.3 Epi-Lipschitz and Prox-regular Sets

This section is devoted to give a summary on two types of special sets, namely, the epi-Lispchitz sets and the prox-regular sets. Epi-Lipschitz sets where first introduced by Rockafellar in 1979 (see [26]) for the finite-dimensional case. The history of prox-regular sets is more complex, but we can trace back their first appearance to 1959, in the celebrated paper of Federer (see [14]), where they were introduced as sets with *positive reach.*

**Definition 4.3.1** (Epi-Lipschitz set) *Let $S$ be a nonempty subset of $X$ and let $x_0 \in S$ and $h \in X \setminus \{0\}$. We say that $S$ is* epi-Lipschitz at $x_0$ in the direction $h$ *if there exists a complement subspace $Z$ of $\mathbb{R}h$, an open neighborhood $W \in \mathcal{N}_X(x_0)$ and a Lipschitz function $f : Z \to \mathbb{R}$ such that*

$$S \cap W = \{z + th \in W \ : \ (z,t) \in Z \times \mathbb{R},\ f(z) \le t\} = L(\operatorname{epi} f) \cap W, \tag{4.13}$$

*where, writing $X = Z \oplus \mathbb{R}h$, $L : Z \times \mathbb{R} \to X$ is the canonic isomorphism given by $L(z,t) = z + th$. We simply say that $S$ is epi-Lipschitz at $x_0$ if there exists a nonzero direction $h \in X \setminus \{0\}$ for which $S$ is epi-Lipschitz at $x_0$ in the direction $h$.*

Note that the name epi-Lipschitz comes from the fact that the set $S$ is locally isomorphic to the epigraph of a Lipschitz function. In 1979, Rockafellar characterized them (in the finite-dimensional case) in terms of the Clarke tangent cone (see [26]). Roughly speaking, he proved that a subset $S$ of $\mathbb{R}^n$ is epi-Lipschitz at a point $x_0$ if and only if $\operatorname{int}(T^C(S;x_0)) \neq \emptyset$. Moreover, the directions on which $S$ is epi-Lipschitz at $x_0$ are precisely the nonzero ones belonging to $\operatorname{int}(T^C(S;x_0))$. This approach still holds for the infinite-dimensional case, replacing $\operatorname{int}(T^C(S;x_0))$ by what is called the *interior tangent cone* of $S$ at $x_0$.

**Definition 4.3.2** (Interior tangent cone) *Let $S$ be a nonempty subset of $X$ and let $x_0 \in S$. We define the* interior tangent cone *of $S$ at $x_0$, denoted by $I(S;x_0)$, as the set of all directions $h \in X$ for which there exist $\varepsilon > 0$ and two neighborhoods $U \in \mathcal{N}_X(x_0)$ and $V \in \mathcal{N}_X(h)$ such that*

$$(U \cap S) + \,]0,\varepsilon[V \subseteq S.$$

**Remark 4.3.3** When introduced, the interior tangent cone $I(S;x_0)$ was called by Clarke and Rockafellar *thehypertangent cone*. The elements belonging to $I(S;x_0)$ were called *hypertangent vectors*. See, e.g., [9, Ch. 2] and [27].

Next proposition summarizes basic properties of the interior tangent cone. The proof is contained in [30], but, for the sake of completeness, we will include it here. Recall that an open cone $C$ is an open set for which

$$\lambda C \subseteq C,\ \forall \lambda > 0.$$

**Proposition 4.3.4** *Let $S \subseteq X$ and $x \in S$. The following hold:*

*(a) $h \in I(S;x)$ if and only if for any sequence $(t_n) \subset \,]0,+\infty[$ converging to $0$, any sequence $(x_n) \subset S$ converging to $x$, and any sequence $(h_n) \subset X$ converging to $h$, we have*

$$x_n + t_n h_n \in S, \quad \textit{(for } n \textit{ large enough).}$$

*(b) $I(S;x) \subseteq T^C(S;x)$ and $I(S;x) + T^C(S;x) = I(S;x)$.*

*(c) $I(S;x)$ is an open convex cone, and whenever $I(S;x) \neq \emptyset$ we have*

$$T^C(S;x) = \mathrm{cl}(I(S;x)) \quad \textit{and} \quad I(S;x) = \mathrm{int}\left(T^C(S;x)\right).$$

*Proof.*

(a) The necessity is direct, and so, we only need to prove the sufficiency. Reasoning by contraposition, fix $h \notin I(S;x)$. Then, for every $n \in \mathbb{N}$, we have that

$$\left(\left(B_X\left(x, \tfrac{1}{n}\right) \cap S\right) + \left]0, \tfrac{1}{n}\right[ B_X\left(h, \tfrac{1}{n}\right)\right) \cap S^c \neq \emptyset.$$

Then, for every $n \in \mathbb{N}$ we can choose $x_n \in B_X\left(x, \frac{1}{n}\right) \cap S$, $t_n \in \left]0, \frac{1}{n}\right[$ and $h_n \in B_X\left(h, \frac{1}{n}\right)$ such that $x_n + t_n h_n \notin S$. Since, by construction, $x_n \to x$, $t_n \to 0$ and $h_n \to h$, we get that the condition of the right of the equivalence doesn't hold for $h$. Therefore, by contraposition, the sufficiency is proved.

(b) The inclusion $I(S;x) \subseteq T^C(S,x)$ holds trivially (use part $(a)$ and Proposition 4.2.13). Now, fix $h \in I(S;x)$ and $h' \in T^C(S;x)$. We want to prove that $d := h + h' \in I(S;x)$. Fix $(x_n) \subset S$ converging to $x$, $(t_n) \subset ]0, +\infty[$ converging to 0 and $(d_n) \subset X$ converging to $d$. We know that there exists a sequence $(h'_n) \subset X$ converging to $h'$ such that $x_n + t_n h'_n \in S$ for $n$ large enough. Consider $y_n := x_n + t_n h'_n$ and $h_n := d_n - h'_n$. Clearly, $y_n \to x$, $h_n \to h$ and $y_n \in S$ for $n$ large enough, which yields, according to the inclusion $h \in I(S;x)$, that

$$x_n + t_n d_n = (x_n + t_n h'_n) + t_n(d_n - h'_n) = y_n + t_n h_n \in S,$$

for all $n$ large enough. Using part $(a)$, we conclude that $h \in I(S;x)$, as we wanted to.

(c) The fact that $I(S;x)$ is an open cone follows from the definition. Also, by part $(b)$, we have that $I(S;x) + I(S;x) \subseteq I(S;x)$, which yields the convexity of $I(S;x)$.

On the other hand, assuming that $I(S;x) \neq \emptyset$, we can choose $h' \in I(S;x)$ and by $(b)$ we have that for every $h \in T^C(S;x)$, the point $h + \frac{1}{n}h' \in I(S;x)$. Then, $T^C(S;x) = \mathrm{cl}(I(S;x))$. Since both sets are convex, this equality also guarantees that $I(S;x) = \mathrm{int}(T^C(S;x))$, finishing the proof.

□

Note that in finite dimension the equality $I(S;x) = \mathrm{int}(T^C(S;x))$ always holds, but, in the infinite-dimensional setting, $I(S;x)$ can be empty even if $\mathrm{int}(T^C(S;x))$ isn't (see [26, Theorem 2 and Counterexample 1]). Nevertheless, if $S$ is convex, then the equality

$$I(S;x) = ]0, +\infty[(\mathrm{int}\, S - x)$$

holds, regardless the dimension of $X$.

For the next theorem, recall that a set $S$ is said to be closed near a point $x \in S$ if there exists an open neighborhood $U \in \mathcal{N}_X(x)$ such that $S \cap U$ is closed, relatively to $U$. Next theorem can be found in the finite-dimensional setting in [26, Theorem 3]. The proof in the general case remains exactly the same.

**Theorem 4.3.5** *Let $S \subseteq X$, $x_0 \in S$ and $h \in X \setminus \{0\}$. Assume that $S$ is closed near $x_0$. The set $S$ is epi-Lipschitz at $x_0$ in the direction $h$ if and only if $h \in I(S; x_0)$.*

*In particular, $S$ is epi-Lipschitz at $x_0$ if and only if $I(S; x_0) \neq \emptyset$.*

**Proposition 4.3.6** *Let $S$ be a closed set of $X$ and let $x_0 \in \operatorname{bd} S$. If $S$ is epi-Lipschitz at $x_0$ then*

$$\{0\} \subsetneq N^C(S; x_0),$$

*that is, there exists a nonzero vector $\xi$ such that $\mathbb{R}_+\{\xi\} \subset N^C(S; x_0)$.*

We finish the introduction of epi-Lipschitz sets establishing a formula for the interior tangent cone of the complement of a set. This proposition can be found in [27] or, more recently, in [13].

**Proposition 4.3.7** *Let $S$ be a nonempty set of $X$ and let $x_0 \in S \cap \operatorname{bd} S$. Then, the equality*

$$I(S^c \cup \{x_0\}; x_0) = -I(S; x_0)$$

*always holds, thus $S$ is epi-Lipschitz at $x_0$ in the direction $h \in X \setminus \{0\}$ if and only if $S^c \cup \{x_0\}$ is epi-Lipschitz in the opposite direction $-h$. So,*

$$T^C(S^c \cup \{x_0\}; x_0) = -T^C(S; x_0),$$

*whenever $S$ is epi-Lipschitz at $x_0$.*

Combining the latter proposition with Proposition 4.2.14(d), we can establish the following direct corollary:

**Corollary 4.3.8** *Let $S$ be a nonempty set of $X$ which is epi-Lipschitz at $x_0 \in S \cap \operatorname{bd} S$. Then,*

$$T^C(\operatorname{bd} S; x_0) = T^C(S; x_0) \cap (-T^C(S; x_0)),$$

*which yields in particular that $T^C(\operatorname{bd} S; x_0)$ is a closed subspace of $X$.*

Now, we will turn to the notion of *prox-regular sets*. The definitions and development included in this part are based mostly on Mazade's Thesis (see [19]), the fundamental work of Poliquin, Rockafellar and Thibault (see [24]) and the nice survey on prox-regular sets of Colombo and Thibault (see [11]). Since the proofs of the propositions in this part are quite technical and require several lemmas, we will omit them, referring the reader to the latter indicated references.

Even though prox-regularity can be defined for general sets, we prefer to settle it only of *closed sets*, since the definition is based on the existence of projections and therefore it only make sense for closed (or locally closed) sets.

**Definition 4.3.9** (Prox-regular set) *Given an extended real $r \in ]0, +\infty]$ and a real $\alpha > 0$, we say that a closed set $S$ of $X$ is* $(r, \alpha)$-prox-regular at $x_0 \in S$ *if for every $x \in S \cap B_X(x_0, \alpha)$ and every $\zeta \in N^P(S; x) \cap \mathbb{B}_X$ we have that*

$$x \in \operatorname{Proj}_S(x + t\zeta), \quad \textit{for every real}\, t \in [0, r]. \tag{4.14}$$

*We say that $S$ is* $r$-prox-regular at $x_0 \in S$ *if it is $(r, \alpha)$-prox-regular at $x_0$ for some $\alpha > 0$ and we simply say that $S$ is* prox-regular at $x_0$ *if there exists $r > 0$ such that $S$ is $r$-prox-regular at $x_0$.*

*Consequently, we say that $S$ is $r$-prox-regular (resp. prox-regular) if it is $r$-prox-regular (resp. prox-regular) at every point $x \in S$.*

It is clear that if $S$ is $(r, \alpha)$-prox-regular at $x_0$, then it is also $(r', \alpha')$-prox-regular at $x_0$ for every $\alpha' \in ]0, \alpha]$ and every $r' \in ]0, r]$.

In the paper [24], Poliquin, Rockafellar and Thibault studied the local prox-regularity of a set $S$. We summarize their results (those that we will need) in the following theorem:

**Theorem 4.3.10** (PRT, 2000) *Let $S$ be a closed set of $X$ and $x_0 \in S$. The following assertions are equivalent:*

*(i) $S$ is prox-regular at $x_0$;*

*(ii) There exists $O \in \mathcal{N}_X(x_0)$ such that $P_S$ is well defined and locally Lipschitz continuous in $O$;*

*(iii) There exist two real constants $\sigma \geq 0$, $\delta > 0$ such that for every $x \in S \cap B_X(x_0, \delta)$ and every $\zeta \in N^P(S; x) \cap \mathbb{B}_X$, one has*

$$\langle \zeta, y - x \rangle \leq \frac{\sigma}{2} \|y - x\|^2, \ \forall y \in S \cap B_X(x_0, \delta);$$

*(iv) There exists $O \in \mathcal{N}_X(x_0)$ such that $d_S$ is continuously differentiable in $O \setminus S$.*

*Moreover, if $S$ is prox-regular at $x_0$, then $S$ is tangentially and normally regular at $x_0$ and there exists a neighborhood $O \in \mathcal{N}_X(x_0)$ for which $P_S$ is well defined in $O$, $d_S$ is Fréchet-differentiable in $O \setminus S$ and its gradient is given by*

$$\nabla d_S(u) = \frac{u - P_S(u)}{d_S(u)}, \ \forall u \in O \setminus S. \tag{4.15}$$

In the Ph.D thesis [19] of M. Mazade, quantified versions are provided for the characterizations of local prox-regularity given in the PRT theorem. To do so, for $r \in ]0,+\infty]$ and $\alpha > 0$ the following local enlargements of the set $S$ at a point $x_0 \in S$ are introduced:

$$\mathcal{R}_S(x_0, r, \alpha) := \left\{x + tv \;:\; x \in S \cap B_X(x_0, \alpha),\ t \in [0, r[,\ v \in N^P(S; x) \cap \mathbb{B}_X\right\}, \quad (4.16)$$
$$\mathcal{W}_S(x_0, r, \alpha) := \{u \in X \;:\; \mathrm{Proj}_S(u) \cap B_X(x_0, \alpha) \neq \emptyset,\ d_S(u) < r\}. \quad (4.17)$$

We summarize the results in [19] that we will use in the following theorem (see [19, Theorem 2.3.3 and Theorem 2.3.4]):

**Theorem 4.3.11** ([19]) *Let $S$ be a closed set of $X$, $x_0 \in S$, $r \in (0,+\infty]$ and $\alpha > 0$. The following assertions are equivalent:*

*(i) $S$ is $(r,\alpha)$-prox-regular at $x_0$;*

*(ii) The set $\mathcal{W}_S(x_0, r, \alpha)$ is open and $P_S$ is well-defined and locally Lipschitz continuous on $\mathcal{W}_S(x_0, r, \alpha)$;*

*(iii) The set $\mathcal{W}_S(x_0, r, \alpha)$ is open and $d_S$ is continuously differentiable on $\mathcal{W}_S(x_0, r, \alpha) \setminus S$ with $\nabla d_S(u) = \frac{u - P_S(u)}{d_S(u)}$ for all $u \in \mathcal{W}_S(x_0, r, \alpha) \setminus S$;*

*(iv) For any $x \in S \cap B(x_0, \alpha)$ and $\zeta \in N^P(S; x)$ one has*

$$\langle \zeta, x' - x\rangle \leq \frac{\|\zeta\|}{2r}\|x' - x\|^2 \quad \text{for all } x' \in S.$$

*Moreover, if $S$ is $(r,\alpha)$-prox-regular at $x_0$, then $\mathcal{R}_S(x_0, r, \alpha)$ and $\mathcal{W}_S(x_0, r, \alpha)$ coincide, and also, for each $\gamma \in ]0,1[$, $P_S(\cdot)$ is Lipschitz continuous on $\mathcal{W}_S(x_0, \gamma r, \alpha)$ with Lipschitz constant $(1-\gamma)^{-1}$.*

It is known (see, e.g., [11]) that $S$ is prox-regular if and only if there exists a continuous function $\rho : S \to (0,+\infty]$ (that we will call *prox-regularity function*) such that for every $x \in S$ and every $\zeta \in N^P(S; x) \cap \mathbb{B}_X$ one has

$$x \in \mathrm{Proj}_S(x + t\zeta), \quad \text{for every real } t \in [0, \rho(x)].$$

It is also known (see, e.g., [11, Chapter 3, Propositions 4 and 11]) that whenever $S$ is $\rho(\cdot)$-prox-regular, the enlargement of $S$

$$U_{\rho(\cdot)}(S) := \{u \in X \;:\; \exists y \in \mathrm{Proj}_S(u) \text{ with } d_S(u) < \rho(y)\}$$

is an open set, $P_S$ is well-defined on $U_{\rho(\cdot)}(S)$ and $d_S^2(\cdot)$ is of class $\mathcal{C}^1$ on $U_{\rho(\cdot)}(S)$.

An interesting property of prox-regular sets is the directional differentiability of the metric projection. This property was developed for convex sets by Zarantonello [31] and later established for prox-regular sets, independently by Canino [7] and Shapiro [29]. Here we state the result as in [11, Theorem 26].

Recall that for a closed set $S$ and a point $x \in \operatorname{bd} S$, the metric projection $P_S$ is said to be directionally differentiable (in the sense of Gâteaux) if there exists a mapping $F : X \to X$ (not necessarily linear) such that

$$P_S(x + th) = x + tF(h) + o(t).$$

In such a case, $F$ is called the *directional derivative* of $P_S$ at $x$.

**Theorem 4.3.12** *Assume that $S$ is a closed prox-regular set at $x \in \operatorname{bd} S$. We have that there exists an open neighborhood $U \in \mathcal{N}_X(x)$ such that $P_S$ is directionally differentiable at every $u \in U \cap S$. Moreover, the directional derivative at $u$ coincides with the metric projection onto $T^B(S; u)$, that is,*

$$P_S(u + th) = P_S(u) + tP_{T^B(S;u)}(h) + o(t).$$

*Proof.* See [11, Theorem 26]. □

## 4.4 Variational properties of submanifolds

In this section we expose some properties of sets which are $\mathcal{C}^p$-submanifolds or which have $\mathcal{C}^p$-smooth boundary (namely, their boundary is a $\mathcal{C}^p$-submanifold). Many of these results are probably known but they are hard to find in the literature. They will form the base of the development of Chapter 5.

We will first introduce the main object which we will work with, namely, the closed bodies.

**Definition 4.4.1** (Closed body) *A subset $S$ of $X$ will be called a closed body (relative to $X$) near $x_0 \in \operatorname{bd} S$ provided there exists an open connected neighborhood $U$ of $x_0$ such that $U \cap S = U \cap \left(\overline{\operatorname{int} S}\right)$ and $U \cap \operatorname{int} S$ is connected.*

*When $U = X$, that is, $S = \overline{\operatorname{int} S}$ and $\operatorname{int} S$ is connected, we will say that $S$ is a* closed body *(relative to $X$).*

Note that, in the latter definition, when $S$ is a closed body near $x_0$, the set $S \cap U$ is in turn connected.

**Proposition 4.4.2** *Let $S \subseteq X$ and $x_0 \in \operatorname{bd} S$ such that $x_0 \in \overline{\operatorname{int} S}$. If $\operatorname{bd} S$ is a $\mathcal{C}^p$-submanifold of $X$ at $x_0$, then $T_{x_0}(\operatorname{bd} S)$ is a closed subspace of $X$ of codimension 1; that is, there exists a closed subspace $Z$ of codimension 1, an open neighborhood $U$ of $x_0$ in $X$ and a $C^p$-diffeormorphism $\varphi : U \to \varphi(U) \subset X$ such that $\varphi(U \cap \operatorname{bd} S) = Z \cap \varphi(U)$.*

*Proof.* Let $U$ be an open neighborhood of $x_0$ such that $M := U \cap \operatorname{bd} S$ is a $\mathcal{C}^p$-submanifold of $X$. Without loss of generality, we may assume that $U$ is connected and also, that there exists a $\mathcal{C}^p$-diffeomorphism $\varphi : U \to X$ and a closed subspace $Z$ of $X$ such that $\varphi(x_0) = 0$ and

$$\varphi(U \cap M) = \varphi(U) \cap Z.$$

Denote $V = \varphi(U)$. As recalled above, we know that $T_{x_0}(\operatorname{bd} S) = T_{x_0} M = D\varphi^{-1}(0)Z$. It is enough to show that $Z$ is a subspace of codimension 1. Assume the contrary and consider two distinct vectors $v_1, v_2 \in V \setminus Z$. Without loss of generality, we may assume that there exists $\delta > 0$ such that $V = B(0, \delta)$. Denoting by $\mathcal{H}_Z$ a Hamel basis of $Z$, we can distinguish two cases:

(I) The set $\mathcal{H}_Z \cup \{v_1, v_2\}$ is linearly independent: Then, for each $t \in [0, 1]$, putting

$$\gamma(t) = tv_1 + (1 - t)v_2 \in V \setminus Z,$$

the mapping $\gamma : [0, 1] \to V \setminus Z$ defines a continuous curve with $\gamma(0) = v_1$ and $\gamma(1) = v_2$.

(II) The set $\mathcal{H}_Z \cup \{v_1, v_2\}$ is not linearly independent: Since $\operatorname{codim}[Z] \geq 2$, there exists $v_3 \in V \setminus Z$ such that both sets $\mathcal{H}_Z \cup \{v_1, v_3\}$ and $\mathcal{H}_Z \cup \{v_2, v_3\}$ are linearly independent. Then, using the latter part, we can construct two continuous curves $\gamma_1 : [0, 1/2] \to V \setminus Z$ and $\gamma_2 : [1/2, 1] \to V \setminus Z$ such that $\gamma_1(0) = v_1$, $\gamma_1(1/2) = \gamma_2(1/2) = v_3$ and $\gamma_2(1) = v_2$. Then, considering the mapping $\gamma : [0, 1] \to V \setminus Z$ given by

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in [0, 1/2] \\ \gamma_2(t) & t \in (1/2, 1], \end{cases}$$

we arrive at the same conclusion as (I).

Since $v_1$ and $v_2$ are two arbitrary distinct points of $V \setminus Z$, the existence of such a continuous curve $\gamma$ entails that $V \setminus Z$ is path-connected, and therefore is connected. Then, since $\varphi^{-1} : V \to U$ is continuous, we derive that $U \setminus M = \varphi^{-1}(V \setminus Z)$ is connected too. This is clearly a contradiction since the two open sets $(\operatorname{int} S) \cap U$ and $S^c \cap U$ are nonempty (according to the assumptions $x_0 \in \overline{\operatorname{int} S}$ and $x_0 \in \operatorname{bd} S$), and they satisfy the equality

$$U \setminus M = \big((\operatorname{int} S) \cap U\big) \cup (S^c \cap U).$$

The proof is therefore complete. □

The next proposition shows that a closed body whose boundary is a $C^p$-submanifold can be represented as the epigraph of a $C^p$-function.

Before proving the proposition we need some features for epi-Lipschitz sets whose boundaries are smooth. So, suppose that $S$ is epi-Lipschitz at $x \in \operatorname{bd} S$ and that $\operatorname{bd} S$ is a

$C^p$-submanifold at $x$. We first note that $T^C(S;x)$ is a half-space. Indeed, by epi-Lipschitz property and Corollary 4.3.8, we know that $\operatorname{int}\big(T^C(S;x)\big) \neq \emptyset$ and

$$T^C(\operatorname{bd} S;x) = T^C(S;x) \cap -T^C(S;x).$$

Taking an orthogonal unit vector $\hat{n}_Z$ of $Z(x) := T_x(\operatorname{bd} S)$, Proposition 4.4.2 and the equality (4.10) of Proposition 4.2.14 tell us that

$$N^C(\operatorname{bd} S;x) = \mathbb{R}\hat{n}(x),$$

where $\hat{n}(x) := D\varphi(x)^*\hat{n}_Z/\|D\varphi(x)^*\hat{n}_Z\|$. It ensues that

$$T^C(S;x) \cap -T^C(S;x) = T^C(\operatorname{bd} S;x) = \{h \in X : \langle \hat{n}(x), h\rangle = 0\}.$$

Since the interior of the closed convex cone $T^C(S;x)$ is nonempty, it results that

$$\text{either } T^C(S;x) = \{h \in X : \langle \hat{n}(x), h\rangle \leq 0\} \quad \text{or} \quad T^C(S;x) = \{h \in X : \langle \hat{n}(x), h\rangle \geq 0\},$$

which confirms that $T^C(S;x)$ is a half-space. We may suppose that $\hat{n}_Z$ is chosen so that the second latter equality holds true. We then derive that

$$N^C(S;x) = \{-t\hat{n}(x) : t \geq 0\}. \tag{4.18}$$

The vector $\hat{n}(x)$ is called the *unit interior normal vector of* $\operatorname{bd} S$ *at* $x$, since it is orthogonal to $Z(x)$ and it "aims" to $\operatorname{int} S$. It is worth noting that $\hat{n}(x)$ doesn't depend on the diffeomorphism nor the model space chosen to describe $\operatorname{bd} S$ as submanifold, since it is fully determined by $Z(x)$ and $T^C(S;x)$. In what follows, we will preserve the notation $Z(x)$ and $\hat{n}(x)$ to denote the tangent space and the unit interior normal vector, respectively.

Note that, if in addition we assume that $S$ is normally regular at $x$ (which is the case, for example, when $S$ is prox-regular at $x$), then

$$N^P(S;x) = \{-t\hat{n}(x) : t \geq 0\}$$

and so, according to Definition 4.2.4, $\operatorname{Ray}_x(S)$ and $\operatorname{Ray}_{x,\lambda}(x)$ (with $\lambda > 0$) are both well-defined.

In view of the proof of the proposition we also state the following simple lemma.

**Lemma 4.4.3** *Let $S$ be a subset of $X$ and $U$ an open set of $X$.*
*(a) The following equalities hold:*

$$\operatorname{int}_U(U \cap S) = U \cap \operatorname{int} S, \quad \operatorname{cl}_U(U \cap S) = U \cap \operatorname{cl} S, \quad \operatorname{bd}_U(U \cap S) = U \cap \operatorname{bd} S.$$

*(b) If $S = \overline{\operatorname{int} S}$, then*

$$U \cap S = \operatorname{cl}_U\big(\operatorname{int}_U(U \cap S)\big) = \operatorname{cl}_U(U \cap \operatorname{int} S).$$

*Proof.* The first two equalities in (a) easily follow from the openness of $U$ and the third is a consequence of the former equalities. Finally, if $S = \overline{\operatorname{int} S}$, then we see from (a) that

$$\mathrm{cl}_U\big(\operatorname{int}_U(U \cap S)\big) = \mathrm{cl}_U(U \cap \operatorname{int} S) = U \cap \mathrm{cl}(\operatorname{int} S) = U \cap S.$$

□

**Proposition 4.4.4** *Let $S \subseteq X$ be a closed body near $x_0 \in \operatorname{bd} S$. Assume that $\operatorname{bd} S$ is a $\mathcal{C}^p$-submanifold at $x_0$ with $p \geq 1$ and denote by $Z(x_0) := T_{x_0}(\operatorname{bd} S)$ the tangent space to the boundary of $S$ at $x_0$. Then $S$ is epi-Lipschitz at $x_0$, and there exist a neighborhood $U_0 \in \mathcal{N}_X(x_0)$ and a function $f : \pi_{Z(x_0)}(U_0) \subseteq Z(x_0) \to \mathbb{R}$ such that $f$ is of class $\mathcal{C}^p$ on $\pi_{Z(x_0)}(U_0)$, $\nabla f(\pi_{Z(x_0)}(x_0)) = 0$ and*

$$U_0 \cap S = \{z + t\hat{n}(x_0) \in U_0 \,:\, z \in Z(x_0), f(z) \leq t\}.$$

*where $\hat{n}(x_0)$ denotes the unit interior normal vector of $\operatorname{bd} S$ at $x_0$. Furthermore, endowing $Z(x_0) \times \mathbb{R}$ with the inner product*

$$\langle (z,t), (z',t') \rangle = \langle z, z' \rangle + tt',$$

*if in addition $S$ is $r$-prox-regular at $x_0$, then $\overline{\operatorname{epi} f}$ is also $r$-prox-regular at $(z_0, f(z_0))$.*

*Proof.* By Proposition 4.4.2, we can choose an open neighborhood $U$ of $x_0$, a $C^p$-diffeomorphism $\varphi : U \to \varphi(U) \subset X$, and a closed subspace $Z$ of $X$ of codimension 1 such that $\varphi(x_0) = 0$ and

$$\varphi(U \cap \operatorname{bd} S) = Z \cap \varphi(U).$$

By replacing $\varphi$ by $D\varphi^{-1}(0) \circ \varphi$, we can choose $Z = Z(x_0)$ and $D\varphi(x_0) = \mathrm{id}_X$. Let $\nu$ be a unit vector of $X$ orthogonal to $Z$. We have that $\mathbb{R}\nu$ is a topological vector subspace complement of $Z$ in $X$, that is, $X = Z \oplus \mathbb{R}\nu$. Noticing that $Z = \{x \in X \,:\, \langle \nu, x \rangle = 0\}$, we see that, for $z + t\nu \in U$ with $z \in Z$ and $t \in \mathbb{R}$,

$$z + t\nu \in U \cap \operatorname{bd} S \Leftrightarrow \langle \varphi(z + t\nu), \nu \rangle = 0.$$

Consider the open set $W := \{(z,t) \in Z \times \mathbb{R} \,:\, z + t\nu \in U\}$ in $Z \times \mathbb{R}$, where $Z$ is equipped with the induced norm, and consider also the $\mathcal{C}^p$ function $F : W \to \mathbb{R}$ defined by

$$F(z,t) := \langle \varphi(z + t\nu), \nu \rangle\,, \quad \text{for all } (z,t) \in W.$$

Write $x_0 = z_0 + t_0\nu$ with $z_0 \in Z$ and $t_0 \in \mathbb{R}$, and note that $F(z_0, t_0) = 0$ and that the derivative with respect to the second variable $t$ at $(z_0, t_0)$ satisfies

$$D_2F(z_0, t_0) = \langle D\varphi(z_0 + t_0\nu)\nu, \nu \rangle = \langle D\varphi(x_0)\nu, \nu \rangle = \|\nu\|^2 = 1.$$

We can apply the implicit function theorem to obtain a connected open neighborhood $Q_0$ of $z_0$ in $Z$, a real $\varepsilon > 0$ and a $\mathcal{C}^p$ function $f : Q_0 \to ]t_0 - \varepsilon, t_0 + \varepsilon[$ such that

$$U_0 := \{z + t\nu : z \in Q_0,\ t \in ]t_0 - \varepsilon, t_0 + \varepsilon[\ \} \subset U$$

and such that, for $z \in Z$ and $t \in \mathbb{R}$

$$\Big(z + t\nu \in U_0 \cap \operatorname{bd} S\Big) \Leftrightarrow \Big(z + t\nu \in U_0 \text{ and } F(z,t) = 0\Big) \Leftrightarrow \Big(z + t\nu \in U_0 \text{ and } t = f(z)\Big).$$

The set $S$ being a closed body near $x_0$, shrinking $Q_0$ and $\varepsilon$ if necessary we may and do suppose that $U_0 \cap \operatorname{int} S$ is connected and $U_0 \cap S = U_0 \cap \big(\overline{\operatorname{int} S}\big)$. Furthermore, for any $h \in Z$ we have

$$\langle \nabla f(z_0), h\rangle = -D_2F(z_0,t_0)^{-1} \circ D_1F(z_0,t_0)h = -D_1F(z_0,t_0)h = -\langle D\varphi(x_0)h, \nu\rangle = 0,$$

since $D_2F(z_0,t_0) = \mathrm{id}_{\mathbb{R}}$ and $D\varphi(x_0)\big|_Z = \mathrm{id}_Z$. Thus, $\nabla f(z_0) = 0$.

With the linear isomorphism $L : Z \times \mathbb{R} \to X$ defined by $L(z,t) := z + t\nu$, clearly $(L^{-1}(U_0)) \cap \operatorname{epi}_s f$ and $(L^{-1}(U_0)) \cap \operatorname{hypo}_s f$ are the two connected components of $L^{-1}(U_0) \setminus \operatorname{gph} f$. It results that $U_0 \cap L(\operatorname{epi}_s f)$ and $U_0 \cap L(\operatorname{hypo}_s f)$ are the two connected components of $U_0 \setminus \operatorname{bd} S$. Since $U_0 \cap \operatorname{int} S$ is a connected component of $U_0 \setminus \operatorname{bd} S$ according to the above lemma, it ensures that either $U_0 \cap \operatorname{int} S = U_0 \cap L(\operatorname{epi}_s f)$ or $U_0 \cap \operatorname{int} S = U_0 \cap L(\operatorname{hypo}_s f)$. Noticing that

$$\begin{aligned} U_0 \cap L(\operatorname{hypo}_s f) &= \{z + t\nu : z \in Q_0,\ t \in ]t_0 - \varepsilon, t_0 + \varepsilon[,\ t < f(z)\} \\ &= \{z + t(-\nu) : z \in Q_0,\ t \in ]-t_0 - \varepsilon, -t_0 + \varepsilon[,\ (-f)(z) < t\}, \end{aligned}$$

and changing $\nu$ by $-\nu$ and $t_0$ by $-t_0$ if necessary, we may suppose that the equality $U_0 \cap \operatorname{int} S = U_0 \cap L(\operatorname{epi}_s f)$ holds true. By the above lemma again we derive that $U_0 \cap S = U_0 \cap L(\operatorname{epi} f)$, which also says that $S$ is epi-Lipschitz at any point in $U_0 \cap S$.

Let us denote $A := L^{-1}$ and endow $Z \times \mathbb{R}$ with the canonical inner product, that is,

$$\langle (z,r), (z',r')\rangle_{Z\times\mathbb{R}} := \langle z, z'\rangle + rr'.$$

Writing any $x \in X$ as $x = \pi_Z(x) + \pi_{\mathbb{R}}(x)\nu$ with $\pi_Z(x) \in Z$ and $\pi_{\mathbb{R}}(x) \in \mathbb{R}$, the bijective linear mapping $A : X \to Z \times \mathbb{R}$ satisfies $A(x) := (\pi_Z(x), \pi_{\mathbb{R}}(x))$ and it is an isomorphism such that $A(U_0 \cap S) = A(U_0) \cap (\operatorname{epi} f)$. Since $f$ is of class $\mathcal{C}^1$, at any $z \in \pi_Z(U_0)$ we have $\partial_C f(z) = \{\nabla f(z)\}$ and therefore

$$N^C\big(\operatorname{epi} f; (z, f(z))\big) = \{\lambda(\nabla f(z), -1) \,:\, \lambda \geq 0\}.$$

Further, taking the linear isomorphism $A$ into account, we have for any $x \in S \cap U_0$ (see again (4.10))

$$N^C(S;x) = N^C(U_0 \cap S; x) = A^*\big(N^C(A(U_0) \cap (\operatorname{epi} f); A(x))\big) = A^*\big(N^C(\operatorname{epi} f; A(x))\big),$$

where $A^*$ denotes the adjoint of $A$. This yields by (4.18)

$$\{-\lambda \hat{n}(x_0) \,:\, \lambda \geq 0\} = N^C(S;x_0) = \{\lambda A^*(\nabla f(z_0), -1) \,:\, \lambda \geq 0\} = \{A^*(0, -\lambda) \,:\, \lambda \geq 0\}.$$

Observing that $A^* = L$, we get that $\hat{n}(x_0) = \nu$, which finishes the first part of the proof.

For the second part of the proof, assume also that $S$ is $r$-prox-regular at $x_0$. Then, by Theorem 4.3.11 there exists $\delta > 0$ such that for all $x \in S \cap B_X(x_0, \delta)$ and every $\xi \in N^P(S; x)$, one has

$$\langle \xi, x' - x \rangle \leq \frac{1}{2r} \|\xi\| \|x' - x\|^2, \ \forall x' \in S.$$

By shrinking $U_0$ if necessary, we may and do assume that $U_0 \subseteq B(x_0, \delta)$. Now, fix $(z, t) \in A(U_0) \cap (\operatorname{epi} f)$ and $\zeta \in N^P\big(\overline{\operatorname{epi} f}; (z, t)\big) \cap \mathbb{B}_X = N^P\big(\operatorname{epi} f; (z, t)\big) \cap \mathbb{B}_X$. For every $(z', t') \in A(U_0) \cap (\operatorname{epi} f)$, we have, by equality (4.6) of Proposition 4.2.7, that

$$\begin{aligned} \langle \zeta, (z', t') - (z, t) \rangle &= \langle (A^*)^{-1} A^* \zeta, (z', t') - (z, t) \rangle \\ &= \langle A^* \zeta, A^{-1}(z', t') - A^{-1}(z, t) \rangle \\ &\leq \frac{1}{2r} \|A^* \zeta\| \|A^{-1}\big((z', t') - (z, t)\big)\|^2 \\ &\leq \frac{1}{2r} \|(z', t') - (z, t)\|^2, \end{aligned}$$

where the last inequality follows from the equalities $A^{-1} = A^*$ and $\|A^*\| = 1$. Now, consider $(z', t') \in (\operatorname{epi} f) \setminus A(U_0)$. Since $z' \in Q_0$ (keep in mind that $f$ is defined only on $Q_0$) and since

$$A(U_0) = Q_0 \times ]t_0 - \varepsilon, t_0 + \varepsilon[$$

we have necessarily that $t' \notin ]t_0 - \varepsilon, t_0 + \varepsilon[$ and in fact, $t' \geq t_0 + \varepsilon > t$ because $t' \geq f(z') > t_0 - \varepsilon$. Since $\max\{t, f(z')\} < t_0 + \varepsilon \leq t'$, we can define $t'' = \max\{t, f(z')\}$ and, noting that $\pi_{\mathbb{R}}(\zeta) \leq 0$ and $(z', t'') \in A(U_0) \cap \operatorname{epi} f$, we can write by what precedes

$$\begin{aligned} \langle \zeta, (z', t') - (z, t) \rangle &= \langle \zeta, (z', t'') - (z, t) \rangle + \langle \zeta, (0, t' - t'') \rangle \\ &\leq \langle \zeta, (z', t'') - (z, t) \rangle \leq \frac{1}{2r} \|(z', t'') - (z, t)\|^2 \\ &= \frac{1}{2r} \big( \langle z' - z, z' - z \rangle + (t'' - t)^2 \big) \\ &\leq \frac{1}{2r} \big( \langle z' - z, z' - z \rangle + (t' - t)^2 \big) = \frac{1}{2r} \|(z', t') - (z, t)\|^2, \end{aligned}$$

where the last inequality is due to the fact that $t \leq t'' < t'$. We then obtain that, for all $(z', t') \in \operatorname{epi} f$

$$\langle \zeta, (z', t') - (z, t) \rangle \leq \frac{1}{2r} \|(z', t') - (z, t)\|^2.$$

Taking limits, we see that the inequality still holds for all $(z', t') \in \overline{\operatorname{epi} f}$. This justifies the $r$-prox-regularity of $\overline{\operatorname{epi} f}$ at $(z_0, f(z_0))$ and finishes the proof. □

In 1984, Poly and Raby proved that, if $X$ is finite-dimensional, then for every closed set $M$ such that it is a $\mathcal{C}^{p+1}$-submanifold at a point $m_0 \in M$, we have that the function $d_M^2(\cdot)$ is of class $\mathcal{C}^{p+1}$ near $m_0$ See [25, Section 1].

To prove this, they use the finite-dimensional assumption only to ensure that $\mathrm{Proj}_S(\cdot)$ is nonempty near $m_0$ (this is provided by the local compactness of $M$). We show that their proof can be directly extended to the Hilbert setting (see Theorem 4.4.6), since the local compactness can be replaced by prox-regularity in order to guarantee the nonemptyness of $\mathrm{Proj}_S(\cdot)$, as the following proposition shows.

**Proposition 4.4.5** *Let $M$ be a closed set of $X$ and let $m_0 \in M$. If $M$ is a $\mathcal{C}^{p+1}$-submanifold at $m_0$, then it is prox-regular at the same point.*

*Proof.* Consider a neighborhood $U$ of $m_0$, a $\mathcal{C}^{p+1}$-diffeomorphism $\varphi : U \to \varphi(U)$ and a closed subspace $Z$ of $X$ such that $\varphi(m_0) = 0$ and $\varphi(U \cap M) = \varphi(U) \cap Z$. Let us denote $S := \overline{\varphi(U) \cap Z}$. Clearly, for $z \in Z \cap \varphi(U)$ we have that $N^C(S; z) = Z^\perp$, so $\langle \xi, z' - z\rangle = 0$ for all $\xi \in N^P(S; z) \subseteq N^C(S; z)$ and $z' \in S$. Theorem 4.3.10(iii) tells us that $S$ is prox-regular at $\varphi(m_0) = 0$.

Choose $\delta > 0$ such that $S' := S \cap B_X(0, \delta) \subseteq \varphi(U)$. We can apply equality (4.10) of Proposition 4.2.14 to get that for each $m \in M \cap \varphi^{-1}(B_X(0, \delta)) = \varphi^{-1}(S')$

$$N^P(\varphi^{-1}(S'); m) \subset N^C(\varphi^{-1}(S'); m) = D\varphi(m)^* \left(N^C(S'; \varphi(m))\right) = D\varphi(m) \left(Z^\perp\right).$$

Shrinking $\delta$ if necessary, we can suppose that $\varphi$ is Lipschitz continuous on $\varphi^{-1}(B_X(0, \delta))$ with constant $\gamma \geq 0$ and that, for all $z, z' \in B_X(0, \delta)$

$$\|\varphi^{-1}(z') - \varphi^{-1}(z) - D\varphi^{-1}(z)(z' - z)\| \leq C\|z' - z\|^2,$$

for some constant $C > 0$. Thus, for any $m, m' \in \varphi^{-1}(S')$ and $\zeta \in N^P(\varphi^{-1}(S'); m) \cap \mathbb{B}_X$ we conclude that

$$\begin{aligned}
\langle \zeta, m' - m\rangle &= \langle \zeta, \varphi^{-1}(\varphi(m')) - \varphi^{-1}(\varphi(m))\rangle \\
&\leq \langle \zeta, D\varphi^{-1}(\varphi(m))(\varphi(m') - \varphi(m))\rangle + \gamma^2 C\|m' - m\|^2 \\
&= \langle (D\varphi(m)^*)^{-1}\zeta, \varphi(m') - \varphi(m)\rangle + \gamma^2 C\|m' - m\|^2 \\
&= \gamma^2 C\|m' - m\|^2,
\end{aligned}$$

which, by Theorem 4.3.10(iii), proves the prox-regularity of $M$ at $m_0$. □

**Theorem 4.4.6** *Let $M$ be a closed subset of a Hilbert space $X$ and let $m_0 \in M$. If $M$ is a $\mathcal{C}^{p+1}$-submanifold at $m_0$ (with $p \geq 1$), then there exists a neighborhood $U \in \mathcal{N}_X(m_0)$ such that*

*(a) $d_M^2(\cdot)$ is of class $\mathcal{C}^{p+1}$ on $U$;*

*(b) $P_M$ is well-defined on $U$ and it is of class $\mathcal{C}^p$ therein.*

*Proof.* Without loss of generality, we will fix $m_0 = 0$. Assume first that $M$ is a $\mathcal{C}^{p+1}$-submanifold at 0. By Proposition 4.1.4, there exist a neighborhood $U \in \mathcal{N}_X(0)$, a neighborhood $V_Z \in \mathcal{N}_Z(0)$, where $Z := T_0M$, and a mapping $\theta : V_Z \to Z^\perp$ of class $\mathcal{C}^{p+1}$ such that $\theta(0) = 0$, $D\theta(0) = 0$ and

$$U \cap M = U \cap L^{-1}(\operatorname{gph}\theta), \tag{4.19}$$

with $L$ defined as in Proposition 4.1.4. Since $M$ is also prox-regular at 0, according to Proposition 4.4.5, we may and do assume by Proposition 4.2.2 that $P_M$ is well-defined on $U$. Let us denote $A := L^{-1}$. It is not hard to see that $L$ and so $A$ are both self-adjoint operators, and so we have that for each $m \in U \cap M$,

$$N^C(M; m) = A^* \left[N^C(\operatorname{gph}\theta, (v, \theta(v)))\right] = A\left[\left\{(-D\theta(v)^* z_2, z_2) \ : \ z_2 \in Z^\perp\right\}\right], \tag{4.20}$$

where $v = \pi_Z(m)$. Let us consider the mapping $\varphi : V_Z \times Z^\perp \to Z \times Z^\perp$ given by

$$\varphi(v, z_2) = (v - D\theta(v)^* z_2, \theta(v) + z_2),$$

and note that $\varphi(0) = 0$. Since by construction $D\varphi(0) = \mathrm{id}_{Z\times Z^\perp}$, we can apply the Local Inverse Function Theorem and therefore there exists an open neighborhood $O = O_1 \times O_2 \in \mathcal{N}_{Z\times Z^\perp}(0)$ included in $L(U)$ and with $O_1 \subseteq V_Z$, such that $\varphi : O \to \varphi(O)$ is a $\mathcal{C}^p$-diffeomorphism.

Now, let $\delta > 0$ small enough such that $B_X(0, 3\delta) \subseteq A(\varphi(O))$. For every $u \in B_X(0, \delta)$ we already know by equation (4.19) and the prox-regularity of $M$ at $P_M(u)$, that there exists $v \in V_Z$ such that

$$P_M(u) = A(v, \theta(v)) \tag{4.21}$$

and then, by the first equality in equation (4.20) and the normal regularity of $M$ at $P_M(u)$,

$$u - P_M(u) \in N^P(M, P_M(u)) = N^C(M, A(v, \theta(v))).$$

Thus, the second equality in equation (4.20) furnishes some $z_2 \in Z^\perp$ such that

$$u - P_M(u) = A(-D\theta(v)^* z_2, z_2). \tag{4.22}$$

Adding the equalities (4.21) and (4.22), we obtain that

$$u = A(v - D\theta(v)^* z_2, \theta(v) + z_2).$$

Since $P_M(u) \in B_X(0, 2\delta)$ then $(A \circ \varphi)^{-1}(u) \in A \circ \varphi)^{-1}(B_X(0, 3\delta))$, and so we conclude that $(v, z_2)$ can be chosen in $O$. By the bijectivity of $\phi := A \circ \varphi$ from $O$ onto $A(\varphi(O))$, we derive that $(v, z_2) = \phi^{-1}(u)$. Finally, using equation (4.21), we can write

$$P_M(u) = A(\pi_Z \circ \phi^{-1}(u), \theta \circ \pi_Z \circ \phi^{-1}(u)),$$

which yields that $P_M$ is of class $\mathcal{C}^p$ on $B_X(0, \delta)$, proving the first implication. □

# Chapter 5

# Smoothness of the Metric Projection onto Nonconvex Bodies in Hilbert spaces

This Chapter is the core of the second part of this thesis. It contains all the new results concerning to the smoothness of the metric projection onto nonconvex sets. We present here a quantified version of Holmes' theorem when the set is a nonconvex body with $C^{p+1}$-smooth boundary. By quantified, we mean that we give an specific neighborhood where the metric projection is $p$ times continuously differentiable, which is determined by the prox-regularity function of the set. We also derive a similar result when the set itself is a $\mathcal{C}^{p+1}$-submanifold.

Finally, based on the work of Fitzpatrick and Phelps [16], we study the converse for Holmes' theorem outside the convex setting.

## 5.1 Smoothness of the metric projection onto nonconvex bodies

Let $S$ be a closed subset of $X$. In order to extend Holmes' theorem to the nonconvex setting we will study the case when $S$ satisfies

(i) $S$ is a closed body relative to the subspace $Y = \overline{\text{aff}}(S)$;

(ii) Considering $S$ as a subset of $Y$, it has a $\mathcal{C}^{p+1}$-smooth boundary.

These hypothesis are based on the fact that the smoothness of the metric projection onto $S$ as a subset of $X$ is fully determined by the smoothness of the metric projection onto $S$

as a subset of $Y$. This observation is formalized in the following proposition.

**Proposition 5.1.1** *Let $S$ be a closed set of $X$ and denote $Y := \overline{\mathrm{aff}}(S)$. Then, for every $x \in X$ we have that*

$$\mathrm{Proj}_S(x) = \mathrm{Proj}_S(\Pi_Y(x)),$$

*and furthermore, defining the sets*

$$O_X = \{x \in X \ :\ P_S \text{ is well-defined and it is of class } \mathcal{C}^p \text{ near } x\}; \text{ and}$$
$$O_Y = \left\{y \in Y \ :\ P_S\big|_Y \text{ is well-defined and it is of class } \mathcal{C}^p \text{ near } y\right\},$$

*we get that $O_X = \Pi_Y^{-1}(O_Y)$.*

*Proof.* Choose $x \in X$. Since $S \subseteq Y$ we can write

$$\|x - v\|^2 = \|x - \Pi_Y(x)\|^2 + \|\Pi_Y^2(x) - v\|^2, \forall v \in S.$$

Therefore, we have that

$$\mathrm{Proj}_S(x) = \mathrm{argmin}\{\|x-v\|^2 \ :\ v \in S\} = \mathrm{argmin}\{\|\Pi_Y(x)-v\|^2 \ :\ v \in S\} = \mathrm{Proj}_S(\Pi_Y(x)),$$

which proves the first part of the theorem. Also, we get that $\Pi_Y^{-1}(O_Y) \subset O_X$. Indeed, for every $x \in \Pi_Y^{-1}(O_Y)$ we have

$$P_S(x) = P_S\big|_Y \circ \Pi_Y(x),$$

and therefore, the inclusion $\Pi_Y^{-1}(O_Y) \subset O_X$ is direct, provided $\Pi_Y$ is a continuous linear operator, and therefore of class $\mathcal{C}^\infty$.

On the other hand, let $x \in O_X$. Noting that for every $y \in Y$ and every $v \in S$ we have that $\Pi_Y(x) + y - v \in Y$, we can write

$$\|x + y - v\|^2 = \|x - \Pi_Y(x)\|^2 + \|\Pi_Y(x) + y - v\|^2$$

and so, we get that $P_S\big|_Y(\Pi_Y(x) + y) = P_S(x + y)$ for every $y \in Y$ such that $x + y \in O_X$, or equivalently

$$P_S\big|_Y(w) = P_S(x + w - \Pi_Y(x)) \quad \text{for all } w \in Y \text{ with } x + w - \Pi_Y(x) \in O_X.$$

By definition, it is clear that $O_X$ is an open set. Choose then a real $\varepsilon > 0$ such that $P_S$ is well defined on $x + B_X(0, \varepsilon)$ and of class $C^p$ therein. Consider the mapping $\ell$ : $\Pi_Y(x) + B_Y(0, \varepsilon) \to X$ defined by $\ell(w) := w + x - \Pi_Y(x)$ for all $w \in \Pi_Y(x) + B_Y(0, \varepsilon)$. Clearly, $\ell\big(\Pi_Y(x)+B_Y(0, \varepsilon)\big) \subset x+B_X(0, \varepsilon)$, hence $P_S\circ\ell$ is well defined on $\Pi_Y(x)+B_Y(0, \varepsilon)$ and of class $C^p$ therein. Since $P_S\big|_Y(w) = (P_S \circ \ell)(w)$ for every $w \in \Pi_Y(x) + B_Y(0, \varepsilon)$, the mapping $P_S\big|_Y$ is of class $C^p$ near $\Pi_Y(x)$. This tells us that $\Pi_Y(x) \in O_Y$, or equivalently $x \in \Pi_Y^{-1}(O_Y)$. We derive that $O_X \subset \Pi_Y^{-1}(O_Y)$, which combined with the previous above inclusion gives the equality $O_X = \Pi_Y^{-1}(O_Y)$. □

Based on the latter proposition, the smoothness of $P_S$ (when it exists) is characterized by the smoothness of $P_S\big|_Y$, and so our target problem can be reduced to prove the extend Holmes' theorem for nonconvex bodies with $\mathcal{C}^{p+1}$-smooth boundary (see Theorem 5.1.6).

### 5.1.1 Extension of Holmes' Theorem

We will first study the case when the set $S$ is the epigraph of a function $f$. The technique to prove Theorem 5.1.2 will be to reduce ourself to this case, helped by Proposition 4.4.4. We recall here equations (4.16) and (4.17): for a set $S$, a point $x_0 \in S$, an extended-real $r \in ]0,+\infty]$ and a positive real $\alpha > 0$ we define, based on Mazade's notation (see [19]), the sets:

$$\mathcal{R}_S(x_0,r,\alpha) := \left\{x + tv \ : \ x \in S \cap B_X(x_0,\alpha),\ t \in [0,r[,\ v \in N^P(S;x) \cap \mathbb{B}_X\right\},$$
$$\mathcal{W}_S(x_0,r,\alpha) := \{u \in X \ : \ \mathrm{Proj}_S(u) \cap B_X(x_0,\alpha) \neq \emptyset,\ d_S(u) < r\}.$$

**Theorem 5.1.2** *Let $O_0 \subseteq X$ be an open set and $f : O_0 \subseteq X \to \mathbb{R}$ be a function of class $\mathcal{C}^{p+1}$ (with $p \geq 1$) near $x_0 \in X$ such that $\nabla f(x_0) = 0$. Assume that $\overline{\mathrm{epi}\, f}$ is $r$-prox-regular at $(x_0, f(x_0))$. For the constant*

$$\lambda = \min\left\{r,\ \left(-2\inf\left\{\langle u, D^2 f(x_0)u\rangle \ : \ u \in \mathbb{B}_X\right\}\right)^{-1}\right\}$$

*there exists an open neighborhood $W$ of $\mathrm{Ray}_{(x_0,f(x_0)),\lambda}(\mathrm{epi}\, f)$ such that*

*(a) $d_{\mathrm{epi}\, f}$ is of class $\mathcal{C}^{p+1}$ on $W$;*

*(b) $P_{\mathrm{epi}\, f}$ is of class $\mathcal{C}^p$ on $W$.*

*Proof.* Let us denote $S = \overline{\mathrm{epi}\, f}$, and $\pi_X : X \times \mathbb{R} \to X$, $\pi_\mathbb{R} : X \times \mathbb{R} \to \mathbb{R}$ the parallel projections associated to the product $X \times \mathbb{R}$. Also, for simplicity, we will write $u = (u_1, u_2)$ for each $u \in X \times \mathbb{R}$. According to the convention $0^{-1} = +\infty$ and noting that $\inf_{u\in\mathbb{B}_X}\langle u, D^2 f(x_0), u\rangle \leq 0$, one sees that $\lambda > 0$.

Since $S$ is $r$-prox-regular at $v_0 := (x_0, f(x_0))$, by Theorem 4.3.11 there exists $\alpha > 0$ small enough for which, by denoting $O := \mathcal{W}_S(v_0, r, \alpha)$, we have that $O$ is open, $\pi_X(O) \subseteq O_0$ (so $O \cap S = O \cap \mathrm{epi}\, f$), $P_S$ is single-valued on $O$, $f$ is of class $\mathcal{C}^{p+1}$ on $\pi_X(O)$, $d_S$ is continuously differentiable in $O \setminus S$ and

$$\nabla d_S(v) = \frac{v - P_S(v)}{d_S(v)},\ \forall v \in O \setminus S. \tag{5.1}$$

Also, since $f$ is of class $\mathcal{C}^{p+1}$, for $x \in \pi_X(O)$ we have that $\partial_P f(x) = \{\nabla f(x)\}$ (see Proposition 4.2.10) and so, by Definition 4.2.8,

$$N^P\big(S; (x, f(x))\big) = \{t(\nabla f(x), -1) \ : \ t \geq 0\}. \tag{5.2}$$

Since $O$ coincides with $\mathcal{R}_S(v_0, r, \alpha)$, we have that for each $v \in S \cap O$

$$P_S\left[\big(v + N^P(S; v)\big) \cap O\right] = v, \tag{5.3}$$

and that $\mathrm{Ray}_{v_0,\lambda}(S) \subseteq O$. Let any $u_0 \in \mathrm{Ray}_{v_0,\lambda}(S)$, and choose three convex neighborhoods $U \in \mathcal{N}_{X\times\mathbb{R}}(u_0)$ and $V, V' \in \mathcal{N}_{X\times\mathbb{R}}(v_0)$ such that

- $U \subseteq O \setminus S$, $V' \subseteq O$, $V \subset V'$;
- $(v_1, f(v_1)) \in V'$ for every $v_1 \in \pi_X(V)$;
- there exists $\delta > 0$ such that $U + (\{0\}\times\, ]-\delta,\delta[\,) \subseteq O \setminus S$; and
- $\operatorname{diam}(\pi_{\mathbb{R}}(V')) < \delta$.

From those assumptions, we have that for each $v \in V$, $(v_1, f(v_1)) \in V'$ and $U - (0, v_2 - f(v_1)) \subseteq O \setminus S$. Let us define the mapping

$$\begin{aligned} F : U \times V &\to X \times \mathbb{R} \\ (u, v) &\mapsto u - v - d_S(u)\varphi(v), \end{aligned}$$

where

$$\varphi(v) = \frac{(\nabla f(v_1), -1)}{\|(\nabla f(v_1), -1)\|} \quad \text{for all } v \in V.$$

We claim that $F(u, v) = 0$ if and only if $v = P_S(u)$. For the sufficiency, let us suppose that $v = P_S(u)$. Then, $u - v \in N^P(S; v)$ and by (5.2) and the definition of $\varphi$, there exists $t \geq 0$ such that

$$u = v + t\varphi(v).$$

Thus, noting that $d_S(u) = \|u - P_S(u)\| = t\|\varphi(v)\| = t$, we conclude that $F(u, v) = 0$. On the other hand, to prove the necessity, let us suppose that $F(u, v) = 0$, so $\|u-v\| = d_S(u)$. Putting $v' = (v_1, f(v_1))$ and noting that $\varphi(v') = \varphi(v)$, we can write

$$u = v + d_S(u)\varphi(v) = v' + d_S(u)\varphi(v') + (0, v_2 - f(v_1)).$$

Therefore, with $u' := u-(0, v_2-f(v_1))$, we have $u'-v' \in N^P(S; v')$ and so, since $v' \in O\cap S$ and $u' \in \big(v' + N^P(S; v')\big) \cap O$, by (5.3) we get $P_S(u') = v'$ and

$$d_S(u') = \|u' - v'\| = \|u - v\| = d_S(u).$$

Define the mapping

$$g : \,]-1, 1 + \delta'[\, \to O \setminus S, \quad \text{given by} \quad g(t) = u - \big(0, t(v_2 - f(v_1))\big),$$

with some $\delta' > 0$ for which $g$ is well-defined. Then,

$$(d_S \circ g)'(t) = -Dd_S(g(t))\big(0, v_2 - f(v_1)\big) = -\pi_{\mathbb{R}}\left(\frac{g(t) - P_S(g(t))}{d_S(g(t))}\right)(v_2 - f(v_1)).$$

Noting that $g(t) - P_S(g(t)) \in N^P\big(S; P_S(g(t))\big)$ and recalling that $g(t) \notin S$, by (5.2) we obtain that $\pi_{\mathbb{R}}\left(\frac{g(t)-P_S(g(t))}{d_S(g(t))}\right) < 0$. Thus, $\operatorname{sgn}((d_S \circ g)'(t)) = \operatorname{sgn}(v_2 - f(v_1))$ for all $t \in \,]-1, 1+\delta'[$ (where $\operatorname{sgn}(\cdot)$ denotes the sign function on $\mathbb{R} \setminus \{0\}$), and we get that if $v_2 \neq f(v_1)$, then

$$(d_S \circ g)(1) \neq (d_S \circ g)(0), \quad \text{that is, } d_S(u') \neq d_S(u),$$

since $d_S \circ g$ is strictly monotone. Since $d_S(u) = d_S(u')$, we conclude that $v_2 = f(v_1)$ and therefore $u = u'$ and $v = v'$. In particular, $P_S(u) = v$, which proves our claim.

We would like now to apply the Implicit Function Theorem to $F$ at $(u_0, v_0)$, so we need to check that

$$D_2F(u_0, v_0) = -id_{X\times\mathbb{R}} - d_S(u_0)\cdot D\varphi(v_0)$$

is an isomorphism. Let us define the mappings $\varphi_1 : (X\times\mathbb{R})\setminus\{0\} \to X\times\mathbb{R}$ and $\varphi_2 : X \to X\times\mathbb{R}$ given by

$$\varphi_1(y) = \frac{y}{\|y\|} \quad \text{and} \quad \varphi_2(x) = (x, -1).$$

We can write $\varphi = \varphi_1 \circ \varphi_2 \circ \nabla f \circ \pi_X$. Recalling that for all $h \in X\times\mathbb{R}$

$$D\varphi_1(y)h = \frac{\|y\|h - \langle \varphi_1(y), h\rangle y}{\|y\|^2}$$

we have that

$$\begin{aligned}
D\varphi(v_0)h &= D\varphi_1((\nabla f(x_0), -1)) \circ D\varphi_2(\nabla f(x_0)) \circ D^2 f(x_0) \circ \pi_X(h)\\
&= D\varphi_1((0,-1))(D^2 f(x_0)h_1, 0)\\
&= \|(0,-1)\|^{-2}\left(\|(0,-1)\|(D^2 f(x_0)h_1, 0) - \left\langle \frac{(0,-1)}{\|(0,-1)\|}, (D^2 f(x_0)h_1, 0)\right\rangle (0,-1)\right)\\
&= (D^2 f(x_0)h_1, 0).
\end{aligned}$$

Thus, $D\varphi(v_0) = (D^2 f(x_0)\circ\pi_X, 0)$. Let us then show that $\mathrm{id}_{X\times\mathbb{R}} + d_S(u_0)D\varphi(v_0)$ is bijective. We may assume that $D^2 f(x_0) \neq 0$, since otherwise the bijectivity is trivial.

**Surjectivity:** Let us consider $h \in X\times\mathbb{R}$ with $h \neq 0$. Since

$$(\mathrm{id}_{X\times\mathbb{R}} + d_S(u_0)D\varphi(v_0))^* h = \mathrm{id}_{X\times\mathbb{R}}(h) + d_S(u_0)(D\varphi(v_0))^* h,$$

it follows that

$$\begin{aligned}
\|(\mathrm{id}_{X\times\mathbb{R}} + d_S(u_0)D\varphi(v_0))^* h\|^2 &= \|h\|^2 + 2d_S(u_0)\langle (D\varphi(v_0))^* h, h\rangle + d_S(u_0)^2\|(D\varphi(v_0))^* h\|^2\\
&= \|h\|^2 + 2d_S(u_0)\langle h_1, D^2 f(x_0)h_1\rangle + d_S(u_0)^2\|(D\varphi(v_0))^* h\|^2\\
&\geq \|h\|^2 + 2d_S(u_0)\left\langle \tfrac{h_1}{\|h\|}, D^2 f(x_0)\tfrac{h_1}{\|h\|}\right\rangle \|h\|^2\\
&\geq \left(1 + 2\inf_{x\in\mathbb{B}_X}\{\langle x, D^2 f(x_0)x\rangle\} d_S(u_0)\right)\cdot\|h\|^2\\
&\geq \left(1 - \frac{1}{\lambda} d_S(u_0)\right)\cdot\|h\|^2,
\end{aligned} \tag{5.4}$$

where the last inequality is due to the definition of $\lambda$. Since $u_0 \in \mathrm{Ray}_{v_0,\lambda}(S) \subset \mathcal{W}_S(v_0, \lambda, \alpha)$, we have that $c = 1 - \lambda^{-1}d_S(u_0) > 0$, and so, by for example [6, Theorem 2.20], the conclusion follows.

**Injectivity:** Let $h \in X \times \mathbb{R}$ such that $\big(\mathrm{id}_{X\times\mathbb{R}} + d_S(u_0)D\varphi(v_0)\big)h = 0$. Then necessarily $h_2 = 0$, provided $\pi_{\mathbb{R}}\left(D\varphi(v_0)h\right) = 0$, and so, recalling that $\inf_{x\in\mathbb{B}_X}\{\langle x, D^2 f(x_0)x\rangle\}$ is less than 0, we can write

$$\begin{aligned} 2\inf_{x\in\mathbb{B}_X}\{\langle x, D^2 f(x_0)x\rangle\}\|h\|^2 &\le \langle h_1, D^2 f(x_0)h_1\rangle = \langle h, (D^2 f(x_0)h_1, 0)\rangle = \langle h, D\varphi(x_0)h\rangle \\ &= d_S(u_0)^{-1}\langle h, d_S(u_0)D\varphi(v_0)h\rangle = -d_S(u_0)^{-1}\|h\|^2, \end{aligned}$$

where the last equality is due to the fact that we have supposed that $h$ belongs to the kernel of $\big(\mathrm{id}_{X\times\mathbb{R}} + d_S(u)D\varphi(v_0)\big)$. But since

$$-d_S(u_0)^{-1} < -\lambda^{-1} \le 2\inf_{x\in\mathbb{B}_X}\{\langle x, D^2 f(x_0)x\rangle\},$$

we have that necessarily $h = 0$, which proves the injectivity.

Now, we can apply the Implicit Function Theorem (IFT) in the following way. Since $d_S$ is of class $\mathcal{C}^1$ in $U$, we have that $F$ is of class $\mathcal{C}^1$ in $U \times V$. Therefore, there exist two neighborhoods $U_1 \in \mathcal{N}(u_0)$ and $V_1 \in \mathcal{N}(v_0)$ and a mapping $\phi : U_1 \to V_1$ such that

(i) $\phi$ is of class $\mathcal{C}^1$;

(ii) For each $u' \in U_1$, $F(u', \phi(u')) = 0$;

(iii) For each $(u', v') \in U_1 \times V_1$, $F(u', v') = 0 \Rightarrow v = \phi(u')$.

Then, by $(ii)$ and $(iii)$ we get that $P_S = \phi$ in $U_1$, and therefore, $P_S$ is of class $\mathcal{C}^1$ on $U_1$, according to $(i)$. Now, looking at formula (5.1), we get that $d_S$ is of class $\mathcal{C}^2$ on $U_1$ and so is $F$ on $U_1 \times V_1$. We can apply recursively this argument as follows:

$$\begin{aligned} d_S \text{ is of class } \mathcal{C}^2 \text{ in } U_1 &\Longrightarrow F \text{ is of class } \mathcal{C}^2 \text{ on } U_1 \times V_1 \\ &\underset{\text{IFT}}{\Longrightarrow} \exists U_2 \in \mathcal{N}(u_0),\ P_S \text{ is of class } \mathcal{C}^2 \text{ on } U_2 \\ &\ \vdots \\ &\Longrightarrow F \text{ is of class } \mathcal{C}^p \text{ on } U_{p-1} \times V_{p-1} \\ &\underset{\text{IFT}}{\Longrightarrow} \exists U_p \in \mathcal{N}(u_0),\ P_S \text{ is of class } \mathcal{C}^p \text{ on } U_p \\ &\Longrightarrow d_S \text{ is of class } \mathcal{C}^{p+1} \text{ on } U_p. \end{aligned}$$

Since $\nabla f$ is of class $\mathcal{C}^p$, the argument ends at this iteration, since we can't ensure that $F$ is of class $\mathcal{C}^{p+1}$. The proof is finished considering $W$ as the union of the $U_p$ obtained by this way for each $u_0 \in \mathrm{Ray}_{v_0,\lambda}(S)$, and noting that $P_S$ and $P_{\mathrm{epi}\, f}$ coincide on $W$ since $W \subseteq O$. $\square$

Observe from the precedent proof that, for the point $u_0 \in \mathrm{Ray}_{v_0,\lambda}(S)$ we have

$$\begin{aligned} DP_S(u_0) &= -[D_2F(u_0,v_0)]^{-1} \circ D_1F(u_0,v_0) \\ &= -[D_2F(u_0,v_0)]^{-1} \circ \left( \mathrm{id}_X - \left\langle \frac{u_0 - P_S(u_0)}{d_s(u_0)}, \cdot \right\rangle \frac{u_0 - P_S(u_0)}{d_S(u_0)} \right) \\ &= -[D_2F(u_0,v_0)]^{-1} \circ \Pi_{X\times\{0\}}. \end{aligned}$$

Also, note that $-D_2F(u_0,v_0)$ maps $X \times \{0\}$ onto $X \times \{0\}$. In particular, we have that $DP_S(u_0)$ restricted to $X \times \{0\}$ is invertible as a function from $X \times \{0\}$ to $X \times \{0\}$.

The following lemma will be crucial in the development below.

**Lemma 5.1.3** *Let $U$ be an open set of $X$ and $f : U \subseteq X \to \mathbb{R}$ be a function of class $\mathcal{C}^{p+1}$ near $x_0 \in X$ such that $\nabla f(x_0) = 0$. Assume that $\overline{\mathrm{epi}\, f}$ is $r$-prox-regular at $(x_0, f(x_0))$. Then, one has*

$$\inf \left\{ \langle u, D^2 f(x_0) u \rangle \ : \ u \in \mathbb{B}_X \right\} \geq -\frac{1}{r}.$$

*Proof.* Let us denote $O := B_{X\times\mathbb{R}}((x_0, f(x_0)), \alpha)$ with $\alpha > 0$ small enough such that $\pi_X(O) \subseteq U$, $f$ is of class $\mathcal{C}^{p+1}$ at $\pi_X(O)$ and $\overline{\mathrm{epi}\, f}$ is $(r,\alpha)$-prox-regular at $(x_0, f(x_0))$. Then, for every $(x,s) \in O \cap \mathrm{epi}\, f$, and every $\xi \in N^P(\mathrm{epi}\, f; (x,s)) = N^P(\overline{\mathrm{epi}\, f}; (x,s))$ we have that

$$\langle \xi, (x', s') - (x,s) \rangle \leq \frac{1}{2r} \|\xi\| \|(x',s') - (x,s)\|^2, \ \forall (x',s') \in \mathrm{epi}\, f. \tag{5.5}$$

Fix $h \in X$. Since for every $x \in \pi_X(O)$, we have that

$$N^P\big(\mathrm{epi}\, f; (x, f(x))\big) = \{t(\nabla f(x), -1) \ : \ t \geq 0\},$$

so using the equality $\nabla f(x_0) = 0$ we can write

$$\begin{aligned} \langle h, D^2 f(x_0) h \rangle &= \lim_{t \searrow 0} \left\langle th, \frac{\nabla f(x_0 + th) - \nabla f(x_0)}{t^2} \right\rangle \\ &= \lim_{t \searrow 0} \left\langle \big(th, f(x_0+th) - f(x_0)\big), \frac{\big(\nabla f(x_0+th), -1\big) - (0,-1)}{t^2} \right\rangle \\ &= \lim_{t \searrow 0} \left\langle \big(x_0 + th, f(x_0+th)\big) - \big(x_0, f(x_0)\big), \frac{\big(\nabla f(x_0+th), -1\big)}{t^2} \right\rangle + \theta(t), \end{aligned}$$

where $\theta(t) := \frac{f(x_0+th) - f(x_0)}{t^2}$. Noting that

$$\theta(t) = \frac{f(x_0+th) - f(x_0) - tDf(x_0)h}{t^2} \ \xrightarrow{t \searrow 0} \ \frac{1}{2} \langle h, D^2 f(x_0) h \rangle,$$

and according to equation (5.5), we can write

$$\begin{aligned}
\langle h, D^2 f(x_0)h\rangle &\geq \lim_{t\searrow 0} -\frac{1}{2rt^2}\|\big(\nabla f(x_0+th),-1\big)\| \, \big\|\big(th, f(x_0+th)-f(x_0)\big)\big\|^2 + \theta(t) \\
&= \lim_{t\searrow 0} -\frac{1}{2r}\|\big(\nabla f(x_0+th),-1\big)\| \left\|\left(h, \frac{f(x_0+th)-f(x_0)}{t}\right)\right\|^2 + \theta(t) \\
&= -\frac{1}{2r}\|\big(\nabla f(x_0),-1\big)\| \, \big\|\big(h, Df(x_0)h\big)\big\|^2 + \lim_{t\searrow 0}\theta(t) \\
&= -\frac{1}{2r}\,\big\|\big(h, Df(x_0)h\big)\big\|^2 + \frac{1}{2}\langle h, D^2 f(x_0)h\rangle = -\frac{1}{2r}\|h\|^2 + \frac{1}{2}\langle h, D^2 f(x_0)h\rangle,
\end{aligned}$$

where the last equality follows from the facts that $Df(x_0)h = 0$ and $\|(h,0)\|^2 = \|h\|^2$. The conclusion follows. □

**Lemma 5.1.4** *Let $S \subseteq X$ be a closed body near $x_0 \in \operatorname{bd} S$. Assume that there exist $r \in ]0,+\infty]$ and $\alpha > 0$ such that $B_X(x_0,\alpha) \cap \operatorname{bd} S$ is a $\mathcal{C}^{p+1}$-submanifold (with $p \geq 1$) and that $S$ is $r$-prox-regular at $x_0$. Then, for $r' = r/2$, there exists a neighborhood $V$ of* $\mathrm{Ray}_{x_0,r'}(S)$ *such that*

- *$d_S$ is of class $\mathcal{C}^{p+1}$ on $V$;*
- *$P_S$ is of class $\mathcal{C}^p$ on $V$.*

*Furthermore, if the set $S$ is $(r,\alpha)$-prox-regular at $x_0$, then*

- *$d_S$ is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_S(x_0,r',\alpha) \setminus S$;*
- *$P_S$ is of class $\mathcal{C}^p$ on $\mathcal{W}_S(x_0,r',\alpha) \setminus S$.*

*Proof.* Shrinking $\alpha$, we may suppose that $S$ is $r$-prox-regular at each point in $B_X(x_0,\alpha) \cap \operatorname{bd} S$. Let $\bar{x} \in B_X(x_0,\alpha) \cap \operatorname{bd} S$. Recalling that $Z(\bar{x}) := T_{\bar{x}}(\operatorname{bd} S)$ and applying Proposition 4.4.4, there exist a neighborhood $U \in \mathcal{N}_X(\bar{x})$ and a function $f : \pi_{Z(\bar{x})}(U) \subseteq Z(\bar{x}) \to \mathbb{R}$ such that, denoting $\bar{z} := \pi_{Z(\bar{x})}(\bar{x})$, $f$ is of class $\mathcal{C}^{p+1}$ in $\pi_{Z(\bar{x})}(U)$, $\nabla f(\bar{z}) = 0$,

$$U \cap S = \{z + t\hat{n}(\bar{x}) \in U \ : z \in Z(\bar{x}),\ f(z) \leq t\},$$

and also, $\overline{\operatorname{epi} f}$ is $r$-prox-regular at $(\bar{z}, f(\bar{z}))$; keep in mind that $\hat{n}(\bar{x})$ denotes the unit interior normal of $\operatorname{bd} S$ at $\bar{x}$. We may and do assume that $U \subseteq B_X(x_0,\alpha)$. By Theorem 5.1.2 and the inequality of Lemma 5.1.3, we have that $P_{\operatorname{epi} f}$ is of class $\mathcal{C}^p$ on a neighborhood $W$ of $\mathrm{Ray}_{(\bar{z},f(\bar{z})),r'}(\operatorname{epi} f)$.

Choose $\delta \in ]0,\alpha[$ small enough such that $B_X(\bar{x},\delta) \subseteq U$ and $S$ is $(r,\delta)$-prox-regular at $\bar{x}$. Let $L : Z(\bar{x}) \times \mathbb{R} \to X$ be the canonic isomorphism given by $L(z,t) = z + t\hat{n}(\bar{x})$. Noting by (4.18) that

$$\begin{aligned}
\mathrm{Ray}_{\bar{x},r'}(S) &= \{\bar{x} - t\hat{n}(\bar{x}) : t \in ]0,r'[\,\} \\
&= L\big(\{(\bar{z}, f(\bar{z})) + t(0,-1) : t \in ]0,r'[\,\}\big) = L\big(\mathrm{Ray}_{(\bar{z},f(\bar{z})),r'}(\operatorname{epi} f)\big),
\end{aligned}$$

we have that $W' := W \cap L^{-1}(\mathcal{W}_S(\bar{x}, r', \delta))$ is also an open neighborhood of $\mathrm{Ray}_{(\bar{z}, f(\bar{z})), r'}(\mathrm{epi}\, f)$. Since $L$ is an isometry, we have that for each $w \in W'$, $P_S(L(w)) \in U \cap S$ and so

$$\begin{aligned}\|L(w) - P_S(L(w))\| &= \|w - L^{-1}\big(P_S(L(w))\big)\\ &\geq \|w - P_{\mathrm{epi}\, f}(w)\| = \|L(w) - L(P_{\mathrm{epi}\, f}(w))\| \geq \|L(w) - P_S(L(w))\|.\end{aligned}$$

Therefore, for each $v \in V_{\bar{x}} := L(W')$, we have that

$$P_S(v) = (L \circ P_{\mathrm{epi}\, f} \circ L^{-1})(v),$$

hence $P_S$ is well-defined on $V_{\bar{x}}$ and it is of class $\mathcal{C}^p$ on $V_{\bar{x}}$. Further, since $W$ can be assumed to be open, the set $V_{\bar{x}}$ is an open neighborhood of $\mathrm{Ray}_{\bar{x}, r'}(S)$, proving the first part of the theorem.

The second part follows directly noting that

$$\mathcal{W}_S(x_0, r', \alpha) \setminus S \subseteq \bigcup \{V_x \ : \ x \in B_X(x_0, \alpha) \cap \mathrm{bd}\, S\},$$

since $\mathcal{W}_S(x_0, r', \alpha)$ and $\mathcal{R}_S(x_0, r', \alpha)$ coincide and since we can write

$$\mathcal{R}_S(x_0, r', \alpha) \setminus S = \bigcup \big\{\mathrm{Ray}_{x, r'}(S) \ : \ x \in B_X(x_0, \alpha) \cap \mathrm{bd}\, S\big\}.$$

□

From the remark after Theorem 5.1.2, we see that, in the proof of the preceding lemma, for each $u_0 \in \mathrm{Ray}_{\bar{x}, r'}(S)$, the operator $DP_{\mathrm{epi}\, f}(L^{-1}(u_0))$ restricted to $Z(\bar{x}) \times \{0\}$ is invertible as a mapping from $Z(\bar{x}) \times \{0\}$ onto $Z(\bar{x}) \times \{0\}$. From this observation, we can conclude that the operator

$$DP_S(u_0) = L \circ DP_{\mathrm{epi}\, f}(L^{-1}(u_0)) \circ L^{-1}$$

restricted to $Z(\bar{x})$ also is invertible as a mapping from $Z(\bar{x})$ onto $Z(\bar{x})$. This yields the following proposition, which will be useful in the study of the converse of Theorem 5.1.6.

**Proposition 5.1.5** *Under the assumptions and notation of Lemma 5.1.4, for each $u_0 \in \mathrm{Ray}_{x_0, r'}(S)$, the operator $DP_S(u_0)$ is invertible as a mapping from $Z(x_0)$ onto $Z(x_0)$.*

*Furthermore, if $S$ is $(r, \alpha)$-prox-regular, then for each $u \in \mathcal{W}_S(x_0, r', \alpha) \setminus S$, the operator $DP_S(u)$ is invertible as a mapping from $Z(P_S(u))$ onto $Z(P_S(u))$.*

We now can proceed to state and prove the extension of Holmes' theorem for nonconvex bodies:

**Theorem 5.1.6** *Let $S \subseteq X$ be a closed body near $x_0 \in \mathrm{bd}\, S$ and let an integer $p \geq 1$. Assume that there exist $r \in ]0, +\infty]$ and $\alpha > 0$ such that $B_X(x_0, \alpha) \cap \mathrm{bd}\, S$ is a $\mathcal{C}^{p+1}$-submanifold and that $S$ is $r$-prox-regular at $x_0$. Then there exists a neighborhood $V$ of $\mathrm{Ray}_{x_0, r}(S)$ such that*

- *$d_S$ is of class $\mathcal{C}^{p+1}$ on $V$;*
- *$P_S$ is of class $\mathcal{C}^p$ on $V$.*

*Furthermore, if the set $S$ is $(r,\alpha)$-prox-regular at $x_0$, then*

- *$d_S$ is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_S(x_0,r,\alpha)\setminus S$;*
- *$P_S$ is of class $\mathcal{C}^p$ on $\mathcal{W}_S(x_0,r,\alpha)\setminus S$.*

*Proof.* Let $U$ be an open connected neighborhood of $x_0$ such that $U\cap\operatorname{int}S$ is connected and $U\cap S = U\cap\left(\overline{\operatorname{int}S}\right)$. Since $S$ is $r$-prox-regular at $x_0$, there exist $\alpha'\in\,]0,\alpha]$ such that $B(x_0,\alpha')\subset U$ and $S$ is $(r,\alpha')$-prox-regular at $x_0$. We will show inductively that for every $n\in\mathbb{N}$, $d_S$ is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_S(x_0,r_n,\alpha')\setminus S$ with $r_n := \sum_{k=1}^{n} 2^{-k}r$. Noting that

$$\operatorname{Ray}_{x_0,r}(S)\subseteq\mathcal{W}(x_0,r,\alpha')\setminus S=\bigcup_{n=1}^{\infty}\mathcal{W}(x_0,r_n,\alpha')\setminus S,$$

and taking into account that $P_S(u)=(u-\nabla d_S(u))/d_S(u)$ for every $u\in\mathcal{W}(x_0,r,\alpha')\setminus S$, proving the latter assertion is enough to conclude the first part of the theorem.

The case $n=1$ is contained in Lemma 5.1.4, so we only need to prove the inductive step. Consider then $n\geq 2$ and assume that $d_S$ is already of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_S(x_0,r_{n-1},\alpha')\setminus S$. It only rests to prove that $d_S$ is of class $\mathcal{C}^{p+1}$ near each point of

$$\mathcal{W}_S(x_0,r_n,\alpha')\setminus\mathcal{W}_S(x_0,r_{n-1},\alpha')=\{u\in\mathcal{W}_S(x_0,r_n,\alpha')\ :\ r_{n-1}\leq d_S(u)<r_n\}.$$

Fix $\bar{u}\in\mathcal{W}_S(x_0,r_n,\alpha')\setminus S$ with $r_{n-1}\leq d_S(\bar{u})<r_n=r_{n-1}+2^{-n}r$. Let us denote $\bar{x}=P_S(\bar{u})$ and choose $\lambda\in\,]0,r_{n-1}[$ such that $d_S(\bar{u})-\lambda<2^{-n}r$. Note by definition of $\mathcal{W}_S(x_0,r_n,\alpha')$ that $\bar{x}\in B(x_0,\alpha')$.

Let us consider the set $S_\lambda := \{x\in X\ :\ d_S(x)\leq\lambda\}$ and the point $\bar{y}:=\bar{x}-\lambda\hat{n}(\bar{x})$ in $\operatorname{bd}S_\lambda$ (where we recall that $\hat{n}(\bar{x})$ denotes the unit interior normal vector of $\operatorname{bd}S$ at $\bar{x}$, which is well-defined since $\operatorname{bd}S$ is a $\mathcal{C}^{p+1}$-submanifold at $\bar{x}$). Since $\lambda<r_{n-1}$ and $\bar{x}\in B(x_0,\alpha')$, we have that $\bar{y}\in\mathcal{W}_S(x_0,r_{n-1},\alpha')$, and so, by hypothesis, $d_S$ is of class $\mathcal{C}^{p+1}$ near $\bar{y}$. Choose $\delta\in\,]0,\alpha'[$ small enough such that $B_X(\bar{y},\delta)\subset\mathcal{W}_S(x_0,r_{n-1},\alpha')\setminus S$. We claim that

$$B_X(\bar{y},\delta)\cap\{d_S<\lambda\}=B_X(\bar{y},\delta)\cap\operatorname{int}(S_\lambda). \tag{5.6}$$

Denoting the second member by $V$ it is clear that it contains the first member. Suppose there is some $u_0\in V$ which is not in the first member. Then $d_S(u_0)=\lambda$, hence $u_0$ is a maximizer of $d_S$ on the open set $V$, which yields $\nabla d_S(u_0)=0$, contradicting the equality $\|\nabla d_S(u_0)\|=1$. The claim is then justified. This says in particular that $B_X(\bar{y},\delta)\cap\operatorname{bd}(S_\lambda)=B_X(\bar{y},\delta)\cap\{d_S=\lambda\}$. Further, $d_S$ is of class $\mathcal{C}^{p+1}$ on $B(\bar{y},\delta)$ and $\nabla d_S(y)$ is surjective from $X$ into $\mathbb{R}$ since $\|\nabla d_S(y)\|=1$ for all $y\in B_X(\bar{y},\delta)$. The set $\operatorname{bd}(S_\lambda)$ is then a $\mathrm{C}^{p+1}$-submanifold of $X$ as seen above for such a level set.

Furthermore, for every $y \in B_X(\bar{y}, \delta) \cap \mathrm{bd}\, S_\lambda$, the $C^{1,1}$-property of $d_S$ near $y$ gives $\partial_P d_S(y) = \{\nabla d_S(y)\}$ and by Proposition 4.2.11 we know that

$$\partial_P d_S(y) = N^P(S_\lambda; y) \cap \mathbb{S}_X,$$

so it follows that

$$N^P(S_\lambda; y) = \{t\nabla d_S(y) \ : \ t \geq 0\} = \{-t\hat{n}(P_S(y)) \ : \ t \geq 0\}. \tag{5.7}$$

Fix $y \in B_X(\bar{y}, \delta) \cap \mathrm{bd}\, S_\lambda$ and denote $x := P_S(y)$. Noting that $\lambda + t < r$ for every $t \in \left]0, \frac{r}{2^{n-1}}\right]$ and recalling that $S$ is $r$-prox-regular at $x$, we have

$$d_S(y - t\hat{n}(x)) = d_S(x - (\lambda + t)\hat{n}(x)) = \lambda + t.$$

Noting also that

$$d_{S_\lambda}(u) = d_S(u) - \lambda, \ \forall u \in X \setminus S_\lambda, \tag{5.8}$$

we can write $d_{S_\lambda}(y - t\hat{n}(x)) = t$, and so $y \in \mathrm{Proj}_{S_\lambda}(y - t\hat{n}(x))$ for every $t \in \left]0, \frac{r}{2^{n-1}}\right]$. In particular, by (5.7), for every $\zeta \in N^P(S_\lambda; y) \cap \mathbb{B}_X$,

$$y \in \mathrm{Proj}_{S_\lambda}(y + t\zeta), \ \forall t \in \left]0, \tfrac{r}{2^{n-1}}\right].$$

Since this last inclusion holds for every $y \in S_\lambda \cap B_X(\bar{y}, \delta)$ (the case of $y \in \mathrm{int}(S_\lambda)$ is trivial since $N^P(S_\lambda; y) = 0$), we conclude that $S_\lambda$ is $\left(\frac{r}{2^{n-1}}, \delta\right)$-prox-regular at $\bar{y}$ according to Definition 4.3.9.

Since $B_X(\bar{y}, \delta) \subseteq \mathcal{W}_S(x_0, r_{n-1}, \alpha')$, we have that for all $y' \in B_X(\bar{y}, \delta)$, $P_S(y') \in U$. We derive that, for any $y' \in B_X(\bar{y}, \delta) \cap \{d_S < \lambda\}$ we have $y' \in P_S(y') + B_X(0, \lambda)$ with $P_S(y') \in S \cap U$, thus

$$B_X(\bar{y}, \delta) \cap \{d_S < \lambda\} = B_X(\bar{y}, \delta) \cap \bigcup_{u \in U \cap S} \big(u + B_X(0, \lambda)\big) = B_X(\bar{y}, \delta) \cap \bigcup_{u \in U \cap \mathrm{int}\, S} \big(u + B_X(0, \lambda)\big), \tag{5.9}$$

where the second equality is due to the fact $U \cap S = U \cap \overline{\mathrm{int}\, S}$. Taking any $y_i$ in the latter set with $i = 1, 2$, there are $x_i \in U \cap \mathrm{int}\, S$ and $b_i \in B_X(0, \lambda)$ such that $y_i = x_i + b_i$. The set $U \cap \mathrm{int}\, S$ being arc-wise connected as an open connected set in the normed space $X$, there exists a continuous mapping $\gamma : [0, 1] \to U \cap \mathrm{int}\, S$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$. The mapping $\gamma_0 : [0, 1] \to B_X(\bar{y}, \delta) \cap \{d_S < \lambda\}$ with $\gamma_0(t) = \gamma(t) + (1 - t)b_1 + tb_2$ is well defined (by (5.9)) and continuous, and further $\gamma_1(0) = y_1$ and $\gamma_0(1) = y_2$. This tells us that the set $B_X(\bar{y}, \delta) \cap \mathrm{int}(S_\lambda) = B_X(\bar{y}, \delta) \cap \{d_S < \lambda\}$ is (arc-wise) connected.

To see that $S_\lambda$ is a closed body near $\bar{y}$ it remains to show that $B_X(\bar{y}, \delta) \cap S_\lambda = B_X(\bar{y}, \delta) \cap \overline{\mathrm{int}(S_\lambda)}$. The second member is obviously included in the first. Take any $y' \in B_X(\bar{y}, \delta)$ with $d_S(y') = \lambda$. Putting $v = \nabla d_S(y') \in \mathbb{S}_X$, for $t > 0$ small enough we have

$$d_S(y' - tv) = d_S(y') - t\big(\langle \nabla d_S(y'), v\rangle + \varepsilon(t)\big) = \lambda - t\big(1 + \varepsilon(t)\big),$$

where $\varepsilon(t) \to 0$ as $t \downarrow 0$, so for $t > 0$ small enough

$$y' - tv \in B_X(\bar{y}, \delta) \cap \{d_S < \lambda\} = B_X(\bar{y}, \delta) \cap \mathrm{int}(S_\lambda),$$

where the equality is due to (5.6). This entails that $y' \in B_X(\bar{y}, \delta) \cap \overline{\operatorname{int}(S_\lambda)}$, hence the desired equality $B_X(\bar{y}, \delta) \cap S_\lambda = B_X(\bar{y}, \delta) \cap \overline{\operatorname{int}(S_\lambda)}$ is justified. The set $S_\lambda$ is then a closed body near $\bar{y}$.

We can then apply the second part of Lemma 5.1.4 to $S_\lambda$ at $\bar{y}$ to get that $d_{S_\lambda}$ is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_{S_\lambda}(\bar{y}, \delta, 2^{-n}r) \setminus S_\lambda$. Finally, by (5.8), we conclude that $d_S$ itself is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_{S_\lambda}(\bar{y}, \delta, 2^{-n}r) \setminus S_\lambda$ and in particular, it is so near $\bar{u}$. The induction (and therefore the proof of the first part of the theorem) is then completed.

The second part of the theorem follows directly from the first one, following the last observations of the proof of Lemma 5.1.4. □

The first corollary is concerned with $\rho(\cdot)$-prox-regular closed bodies.

**Corollary 5.1.7** *Let $S \subseteq X$ be a closed body such that* $\operatorname{bd} S$ *is a $\mathcal{C}^{p+1}$-submanifold with $p \geq 1$. If $S$ is $\rho(\cdot)$-prox-regular, then*

- *$d_S$ is of class $\mathcal{C}^{p+1}$ on $U_{\rho(\cdot)}(S) \setminus S$;*
- *$P_S$ is of class $\mathcal{C}^p$ on $U_{\rho(\cdot)}(S) \setminus S$.*

*Proof.* Fix $u \in U := U_{\rho(\cdot)}(S) \setminus S$. Since $S$ is $\rho(\cdot)$-prox-regular, we have that there exists $y \in \operatorname{Proj}_S(u)$ such that $d_S(u) < \rho(y)$. Let us fix a real $r$ with $d_S(u) < r < \rho(y)$. Since $\rho$ is continuous, there exists a neighborhood $V \in \mathcal{N}_X(y)$ on which $\rho(v) > r$ for each $v \in S \cap V$. Therefore, by properties related to $U_{\rho(\cdot)}(S)$ recalled in Section 4.3 and by Theorem 4.3.11 the set $S$ is $r$-prox-regular at $y$. Then, by Theorem 5.1.6 there exists $\alpha > 0$ small enough such that $P_S$ is well-defined on $\mathcal{W}_S(y, r, \alpha) \setminus S$ and it is of class $\mathcal{C}^p$ on this open set. Noting that

$$u \in (\mathcal{W}_S(y, r, \alpha) \setminus S) \cap U \subseteq U,$$

and that both sets $\mathcal{W}_S(y, r, \alpha) \setminus S$ and $U$ are open, we conclude that $P_S$ is well-defined near $u$ and it is of class $\mathcal{C}^p$ near $u$. Since $u$ is arbitrary, the conclusion follows. □

In the case that $S$ is a convex body, recalling that all convex closed sets are $(+\infty)$-prox-regular, we can recuperate Holmes' theorem as corollary of Theorem 5.1.6:

**Corollary 5.1.8** *[Holmes, 1973] Let $K \subseteq X$ be a convex body and suppose that* $\operatorname{bd} K$ *is a $\mathcal{C}^{p+1}$-submanifold at a point $x_0 \in \operatorname{bd} K$, with $p \geq 1$. Then there exists an open neighborhood $W$ of* $\operatorname{Ray}_{x_0}(K)$ *such that*

- *$d_S$ is of class $\mathcal{C}^{p+1}$ on $W$;*
- *$P_S$ is of class $\mathcal{C}^p$ on $W$.*

### 5.1.2 Smoothness of the metric projection onto submanifolds

Based on Theorem 4.4.6, we will prove an analogous version of Theorem 5.1.6 when $S$ is itself a $\mathcal{C}^{p+1}$-submanifold, instead of a nonconvex body with $\mathcal{C}^{p+1}$-smooth boundary.

**Lemma 5.1.9** *Let $M$ be a closed set of $X$ such that $M$ is a $\mathcal{C}^1$-submanifold at $m_0 \in M$. If $M$ is $r$-prox-regular at $m_0$, then for every $\lambda \in ]0, r[$, the set*

$$M_\lambda := \{x \in X \ : \ d_M(x) \leq \lambda\}$$

*is a closed body near each point $y_0 := m_0 + \lambda v$, where $v \in N^P(M; m_0) \cap \mathbb{S}_X$, and there exists $\delta > 0$ such that*

$$B_X(y_0, \delta) \cap \{d_M < \lambda\} = B_X(y_0, \delta) \cap \mathrm{int}(M_\lambda) \quad \textit{and} \quad B_X(y_0, \delta) \cap M_\lambda = B_X(y_0, \delta) \cap \overline{\mathrm{int}(M_\lambda)}.$$

*Proof.* Note first that $y_0 \in M_\lambda$ and $d_M(y_0) = \lambda$ by the $r$-prox-regularity of $M$. Since $M$ is a $\mathcal{C}^1$-submanifold at $m_0$, there exist a closed subspace $Z$ of $X$, an open convex neighborhood $U \in \mathcal{N}(0)$ and a $C^1$-diffeomorphism $\varphi : U \to \varphi(U)$ such that $\varphi(0) = m_0$ and

$$\varphi(U \cap Z) = \varphi(U) \cap M.$$

Since $U \cap Z$ is arc-wise connected (as a convex set), we get that $\varphi(U) \cap M$ is also arc-wise connected. Now, choose $\alpha, \delta > 0$ small enough such that $B_X(m_0, \alpha) \subseteq \varphi(U)$, $M$ is $(r, \alpha)$-prox-regular at $m_0$ and $B_X(y_0, \delta) \subseteq \mathcal{W}_M(m_0, r, \alpha)$. As in the proof of Theorem 5.1.6, we have that

$$B_X(y_0, \delta) \cap \{d_M < \lambda\} = B_X(y_0, \delta) \cap \mathrm{int}(M_\lambda).$$

Now, since for each $y \in B_X(y_0, \delta)$ we have that $P_M(y) \in \varphi(U)$, we can write

$$B_X(y_0, \delta) \cap \{d_M < \lambda\} = B_X(y_0, \delta) \cap \bigcup_{u \in \varphi(U) \cap M} (u + B_X(0, \lambda)). \tag{5.10}$$

Taking any $y_1, y_2 \in B_X(y_0, \delta) \cap \{d_M < \lambda\}$, we can find $m_1, m_2 \in \varphi(U) \cap M$ and $b_1, b_2 \in B_X(0, \lambda)$ such that $y_i = m_i + b_i$ for $i = 1, 2$. Since $\varphi(U) \cap M$ is arc-wise connected, there exists a continuous mapping $\gamma : [0, 1] \to \varphi(U) \cap M$ with $\gamma(0) = m_1$ and $\gamma(1) = m_2$. Thus, the mapping $\gamma_0 : [0, 1] \to B_X(y_0, \delta) \cap \{d_M < \lambda\}$ given by $\gamma_0(t) = \gamma(t) + (1 - t)b_1 + tb_2$ is well-defined by (5.10), is continuous, $\gamma_0(0) = y_1$ and $\gamma_0(1) = y_2$. We get that the set $B_X(y_0, \delta) \cap \{d_M < \lambda\} = B_X(y_0, \delta) \cap \mathrm{int}(M_\lambda)$ is therefore (arc-wise) connected.

We can show that $B_X(y_0, \delta) \cap M_\lambda = B_X(y_0, \delta) \cap \overline{\mathrm{int}(M_\lambda)}$ following the same argument as in the end of the proof of Theorem 5.1.6. The proof is now complete. □

**Theorem 5.1.10** *Let $M$ be a closed set of $X$ which is a $\mathcal{C}^{p+1}$-submanifold at $m_0 \in M$ with $p \geq 1$. If $M$ is $r$-prox-regular at $m_0$, then there exists $\alpha > 0$ such that*

- *$d_M^2(\cdot)$ is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_M(m_0, r, \alpha)$;*

- $P_M$ *is of class* $\mathcal{C}^p$ *on* $\mathcal{W}_M(m_0, r, \alpha)$.

*Proof.* By Theorem 4.4.6, there exists $\varepsilon > 0$ small enough such that $d_M^2(\cdot)$ is of class $\mathcal{C}^{p+1}$ on $B_X(m_0, \varepsilon)$. Choose then $\alpha \in ]0, \varepsilon[$ such that $M$ is $(r, \alpha)$-prox-regular at $m_0$ and choose also $\lambda \in ]0, r[$ such that $\alpha + \lambda < \varepsilon$. In particular, we have that $\overline{\mathcal{W}_M(m_0, \lambda, \alpha)} \subseteq B_X(m_0, \varepsilon)$. Fix $u \in \mathcal{W}_M(m_0, r, \alpha) \setminus B_X(m_0, \varepsilon)$. We have that $\lambda < d_M(u) < r$.

By definition of $\mathcal{W}_M(m_0, r, \alpha)$ we can take $m \in B_X(m_0, \alpha) \cap M$ and $\nu \in N^P(M; m) \cap \mathbb{S}_X$ such that $u = m + d_M(u)\nu$. Put $y = m + \lambda\nu$. Defining $M_\lambda$ as in Lemma 5.1.9 we have that $y \in \operatorname{bd} M_\lambda$ and $M_\lambda$ is a closed body near $y$, and for some real $\delta > 0$ we have $B_X(y, \delta) \subset \mathcal{W}_M(m_0, r, \alpha)$ along with

$$B_X(y, \delta) \cap \operatorname{bd} M_\lambda = B_X(y, \delta) \cap \{d_M = \lambda\} \quad \text{and} \quad B_X(y, \delta) \cap M_\lambda = B_X(y, \delta) \cap \overline{\operatorname{int}(M_\lambda)}.$$

Fix any $y' \in B_X(y, \delta) \cap \operatorname{bd} M_\lambda$. By the remarks preceding the proof of Theorem 5.1.6, we also know that $\operatorname{bd} M_\lambda$ is a $\mathcal{C}^{p+1}$-submanifold at $y'$, since $d_M$ is of class $\mathcal{C}^{p+1}$ near $y'$ with $\nabla d_M(y') \neq 0$. Further, by Proposition 4.2.11, we have that

$$\{\nabla d_M(y')\} = \partial_P d_M(y') = N^P(M_\lambda; y') \cap \mathbb{S}_X,$$

so setting $\nu' := \nabla d_M(y')$, it follows that $N^P(M_\lambda; y') = \{t\nu' \ : \ t \geq 0\}$. Note that, setting $m' := P_M(y')$ so we can write $y' = m' + \lambda\nu'$, and hence

$$d_M(y' + t\nu') = d_M(m' + (t + \lambda)\nu') = t + \lambda, \ \forall t \in [0, r - \lambda[.$$

Also, noting that

$$d_{M_\lambda}(x) = d_M(x) - \lambda, \ \forall x \in X \setminus M_\lambda, \tag{5.11}$$

we conclude that $d_{M_\lambda}(y' + t\nu') = t$ for every $t \in [0, r - \lambda[$. In particular, fixing $r' \in ]d_M(u) - \lambda, r - \lambda[$, we have that for all $y' \in B_X(y, \delta) \cap \operatorname{bd} M_\lambda$ and $\zeta \in N^P(M_\lambda; y') \cap \mathbb{B}_X$,

$$y' \in \operatorname{Proj}_{M_\lambda}(y' + t\zeta), \ \forall t \in [0, r'],$$

and so, $M_\lambda$ is $(r', \delta)$-prox-regular at $y$. Applying Theorem 5.1.6 it results that, for $\alpha' := \delta$, the function $d_{M_\lambda}$ is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_{M_\lambda}(y, r', \alpha') \setminus M_\lambda$. By equation (5.11) and since $u \in \mathcal{W}_{M_\lambda}(y, r', \alpha') \setminus M_\lambda$, it ensues that $d_M(\cdot)$ (and therefore $d_M^2(\cdot)$) is of class $\mathcal{C}^{p+1}$ near $u$. Since the function $d_M^2$ is also of class $\mathcal{C}^{p+1}$ on $B_X(m_0, \varepsilon)$, we conclude that it is of class $\mathcal{C}^{p+1}$ on the whole open set $\mathcal{W}_M(m_0, r, \alpha)$. □

Observing the proof of Corollary 5.1.7, we can establish the following direct result from Theorem 5.1.10:

**Corollary 5.1.11** *Let $M$ be a closed set of $X$. Assume that $M$ is a $\mathcal{C}^{p+1}$-submanifold. If $M$ is $\rho(\cdot)$-prox-regular, then*

- $d_M^2(\cdot)$ *is of class* $\mathcal{C}^{p+1}$ *on* $U_{\rho(\cdot)}(M)$*;*

- $P_M$ *is of class* $\mathcal{C}^p$ *on* $U_{\rho(\cdot)}(M)$.

## 5.2 Converse of the extension of Holmes' Theorem

In all this section, we will work with a closed set $S$. For a point $x \in X \setminus S$ such that $P_S(x)$ exists, we will denote

$$H_S[x] := \{h \in X \ : \ \langle x - P_S(x), h\rangle = 0\}, \tag{5.12}$$

namely, $H_S[x]$ corresponds to the hyperplane orthogonal to $x - P_S(x)$. If there is no confusion, we will simply write $H[x]$ instead of $H_S[x]$. Here we follow the strategy of Fitzpatrick and Phelps (see [16]).

We will need also the notion of partial derivatives. Let $Y$ be another Hilbert space and $Z$ be a closed subspace of $X$. We say that a continuous mapping $F : X \to Y$ is *partially Gâteaux-differentiable* (*partially G-differentiable*, for short) at a point $x \in X$ with respect to $Z$ if there exists a continuous linear operator $A \in \mathcal{L}(Z; Y)$ such that we can write

$$F(x + th) = F(x) + tA(h) + o(t), \tag{5.13}$$

for every direction $h \in Z$. In such a case, we call $A$ the partial G-derivative of $F$ at $x$ with respect to $Z$ and we denote it by $D_{G,Z}F(x)$.

Analogously, we say that $F : X \to Y$ is *partially Fréchet-differentiable* (*partially F-differentiable*, for short) at a point $x \in X$ with respect to $Z$ if there exists a continuous linear operator $A \in \mathcal{L}(Z; Y)$ such that we can write

$$F(x + h) = F(x) + A(h) + o(h), \tag{5.14}$$

for every direction $h \in Z$. In such a case, we call $A$ the partial F-derivative of $F$ at $x$ with respect to $Z$ and we denote it by $D_{F,Z}F(x)$.

### 5.2.1 Properties of the derivatives of the metric projection

In this section, we will present some properties of the Gâteaux derivative of the metric projection $P_S$ when it exists. They will be very useful in the development of the proof of the converse of Theorem 5.1.6. We start with the symmetric positive property of the Gâteaux derivative of the metric projection.

**Lemma 5.2.1** *Let $S$ be a closed set of $X$ and let $x \in X \setminus S$. Assume that $P_S(\cdot)$ is well-defined and continuous on a neighborhood of $x$. If $D_G P_S(x)$ exists, then it coincides with the second (Gâteaux) derivative at $x$ of the convex function $\Psi : X \to \mathbb{R}$ given by*

$$\Psi(h) := \tfrac{1}{2}\|h\|^2 - \tfrac{1}{2}d_S^2(h), \ \forall h \in X.$$

*Consequently, $D_G P_S(x)$ is a symmetric and positive operator.*

*Proof.* On one hand, the function $\Psi$ is obviously continuous. On the other one, as observed by Asplund in [2], we have that

$$\begin{aligned}\Psi(h) &= \tfrac{1}{2}\|h\|^2 - \inf_{y\in S}\{\tfrac{1}{2}\|h-y\|^2\} \\ &= \tfrac{1}{2}\|h\|^2 - \tfrac{1}{2}\|h\|^2 - \inf_{s\in S}\{\langle h,-y\rangle + \tfrac{1}{2}\|y\|^2\} \\ &= \sup_{y\in S}\{\langle h,y\rangle - \tfrac{1}{2}\|y\|^2\}.\end{aligned}$$

Therefore, $\Psi$ can be written as the supremum of affine functions, which entails its convexity. Now, let $U \in \mathcal{N}_X(x)$ be an open neighborhood on which $P_S(\cdot)$ is well-defined and continuous. Fix $u \in U$ and $u^* \in \partial\Psi(u)$. We have that for all $u' \in U$,

$$\begin{aligned}\langle u^*, u'-u\rangle &\leq \Psi(u') - \Psi(u) \\ &= \tfrac{1}{2}\|u'\|^2 - \tfrac{1}{2}d_S^2(u') - \tfrac{1}{2}\|u\|^2 + \tfrac{1}{2}d_S^2(u) \\ &\leq \tfrac{1}{2}\|u'\|^2 - \tfrac{1}{2}\|u\|^2 - \tfrac{1}{2}\|(u'-u) + (u - P_S(u'))\|^2 + \tfrac{1}{2}\|u - P_S(u')\|^2 \\ &= \tfrac{1}{2}\|u'\|^2 - \tfrac{1}{2}\|u\|^2 - \tfrac{1}{2}\|u'-u\|^2 - \langle u'-u, u-P_S(u')\rangle \\ &= \langle u'-u, P_S(u')\rangle.\end{aligned}$$

Now, taking $h \in X$ and replacing $u'$ by $u+th$ (with $t>0$ small enough) in the latter development, we can write

$$\langle u^*, h\rangle = \lim_{t\searrow 0}\frac{\langle u^*, th\rangle}{t} \leq \lim_{t\searrow 0}\frac{\langle P_S(u+th), th\rangle}{t} = \langle P_S(u), h\rangle.$$

This entails that $u^* = P_S(u)$ and so $\partial\Psi(u) = \{P_S(u)\}$, since the continuity of $\Psi$ ensues the nonemptyness of $\partial\Psi(u)$. We get therefore that $\Psi$ is G-differentiable on $U$ with $\nabla\Psi(u) = P_S(u)$, for all $u \in U$. In particular, since $P_S$ is continuous, we get that $\Psi$ is of class $\mathcal{C}^1$ with $D_F\Psi(u) = \langle P_S(u), \cdot\rangle$, for all $u \in U$.

Now, since $x \in U$, we conclude that $D_G P_S(x)$ is the second (Gâteaux) derivative of $\Psi$ at $x$, and so, by convexity of $\Psi$, we get that $D_G P_S(x)$ is a symmetric and positive operator, finishing the proof. □

The following proposition shows that $D_G P_S(x)$ can be regarded as a linear mapping from $H[x]$ into $H[x]$, since $D_G P_S(x)$ can be (left and right) composed with $\Pi_{H[x]}$ without being modified.

**Proposition 5.2.2** *Let $S$ be a closed subset of $X$ and fix $x \in X \setminus S$. Assume that $P_S$ is well defined and continuous on a neighborhood of $x$. If $D_G P_S(x)$ exists, then*

$$D_G P_S(x) \circ \Pi_{H[x]} = D_G P_S(x) = \Pi_{H[x]} \circ D_G P_S(x). \tag{5.15}$$

*In particular, $D_G P_S(x)X \subseteq H[x]$.*

*Proof.* Fix $y \in X$. Since $y - \Pi_{H[x]}(y) = \lambda(x - P_S(x))$ for some $\lambda \in \mathbb{R}$, it is enough, in order to get that $D_G P_S(x) \circ \Pi_{H[x]} = D_G P_S(x)$, to show that $D_G P_S(x)(x - P_S(x)) = 0$. By Proposition 4.2.2, we see that $P_S(z) = P_S(x)$ for every $z \in [P_S(x), x]$, and so

$$D_G P_S(x)(x - P_S(x)) = \lim_{t \searrow 0} \frac{P_S(x + t(x - P_S(x))) - P_S(x)}{t} = 0.$$

Since $D_G P_S(x)$ and $\Pi_{H[x]}$ are both symmetric operators and since the composition $D_G P_S(x) \circ \Pi_{H[x]}$ is also symmetric (by the preceding development), we can write

$$\langle \Pi_{H[x]} \circ D_G P_S(x) h, h' \rangle = \langle h, D_G P_S(x) \circ \Pi_{H[x]} h' \rangle = \langle D_G P_S(x) \circ \Pi_{H[x]} h, h' \rangle,$$

for every $h, h' \in X$. We conclude that $D_G P_S(x)$ and $\Pi_{H[x]}$ commute, which finishes the proof. □

**Lemma 5.2.3** *Let $S$ be a closed set of $X$ and let $x \in X \setminus S$. Assume that $P_S(x)$ exists and that $S$ is $(r, \alpha)$-prox-regular at $P_S(x)$ for some $\alpha > 0$ and some $r \in ]0, +\infty]$ with $d_S(x) < r$. If $D_G P_S(x)$ exists, then*

$$\|D_G P_S(x)\| \leq \frac{1}{1 - r^{-1} d_S(x)}.$$

*In particular, for any $\delta \in ]0, 1 - r^{-1} d_S(x)[$, we have that the operator $\mathrm{id}_X - \delta D_G P_S(x)$ is positive and invertible.*

*Proof.* Without loss of generality, we may assume that $r < +\infty$, since if the statement of the lemma holds for every $r \in ]0, +\infty[$, then the result is completed by taking $r \to +\infty$. So, fix $r \in ]0, +\infty[$.

Consider $\gamma \in ]0, 1[$ such that $d_S(x) < \gamma r$. We then have that $x \in \mathcal{W}_S(P_S(x), \gamma r, \alpha)$ and by Theorem 4.3.11, we know that $P_S(\cdot)$ is $(1 - \gamma)^{-1}$-Lipschitz continuous in this set. Then, we get that

$$\begin{aligned} \|D_G P_S(x) h\| &\leq t^{-1} \left( \|P_S(x + th) - P_S(x)\| + o(t) \right) \\ &\leq t^{-1} \left( (1 - \gamma)^{-1} \|th\| + o(t) \right) \xrightarrow{t \downarrow 0} (1 - \gamma)^{-1} \|h\| \end{aligned}$$

Thus, $\|D_G P_S(x)\| \leq (1 - \gamma)^{-1}$. Now we can take $\gamma \downarrow r^{-1} d_S(x)$ and so, taking limit, we get that

$$\|D_G P_S(x)\| \leq (1 - r^{-1} d_S(x))^{-1} = \frac{r}{r - d_S(x)}.$$

Now, note that for each $\delta \in \left] 0, \frac{r - d_S(x)}{r} \right[$ we get that

$$\| \mathrm{id}_X - (\mathrm{id}_X - \delta D_G P_S(x)) \| = \delta \|D_G P_S(x)\| < 1,$$

and so, $\mathrm{id}_X - \delta D_G P_S(x)$ is positive and invertible, since every operator in $B_{\mathcal{L}(X)}(\mathrm{id}_X, 1)$ enjoys the property. □

**Proposition 5.2.4** *Let $S$ be a closed set of $X$ and let $x \in X \setminus S$. Assume that $P_S(x)$ exists and that $S$ is $(r,\alpha)$-prox-regular at $P_S(x)$ for some $\alpha > 0$ and some $r \in ]0,+\infty]$ with $d_S(x) < r$. Then, for every $y \in \mathrm{Ray}_{P_S(x),r}(S)$ the following hold:*

*(a) If $D_G P_S(x)$ (resp. $D_F P_S(x)$) exists, then $D_G P_S(y)$ (resp. $D_F P_S(y)$) also exists. Furthermore, if $D_G P_S(x)\big|_{H[x]}$ is injective over $H[x]$ (resp. surjective onto $H[x]$), then $D_G P_S(y)\big|_{H[y]}$ is also injective over $H[y] = H[x]$ (resp. surjective onto $H[y] = H[x]$).*

*(b) If $P_S(\cdot)$ is of class $\mathcal{C}^p$ (with $p \geq 1$) near $x$, then it is also of class $\mathcal{C}^p$ near $y$.*

*Proof.* Without lose of generality, we may and do suppose that $r < +\infty$. Choose $\gamma \in ]0,1[$ such that $\max\{d_S(x), d_S(y)\} < \gamma r$. We know that $P_S(\cdot)$ is Lipschitz continuous on $\mathcal{W}_S(P_S(x), \gamma r, \alpha)$ with Lipschitz constant $K := (1-\gamma)^{-1}$. In particular, $\|D_G P_S(x)\| \leq K$. Now, choose a sequence $\{y_i\}_{i=0}^n \subset [x,y]$ such that $y_0 = x$, $y_n = y$ and

$$y_{i+1} = y_i + t_i(y_i - P_S(y_i))$$

with $t_i \in \mathbb{R}$ small enough such that $|t_i| < (1+K)^{-1}$. Observe that, since $d_S(y_i + t_i(y_i - P_S(y_i))) = d_S(y_{i+1}) < \gamma r$, there exists $\delta_i > 0$ small enough satisfying

$$y_i + h + t_i(y_i + h - P_S(y_i + h)) \in \mathcal{W}_S(P_S(x), \gamma r, \alpha),\ \forall h \in B_X(0, \delta_i). \tag{5.16}$$

Noting that $P_S(y_i) = P_S(x)$ and that $\|D_G P_S(y_i)\| \leq K$ for all $i \in \{0, \ldots, n\}$, it is enough to show that both statements $(a)$ and $(b)$ hold replacing $y$ by $y_1$. The general case arrives inductively, replacing in the $i$th step the roles of $x$ and $y$ by $y_{i-1}$ and $y_i$ respectively.

Assume then that $n = 1$ and denote $t := t_1$. Let us observe first that the operator $A := \mathrm{id}_X + t(\mathrm{id}_X - D_G P_S(x))$ is invertible. Indeed, we have that

$$\|\mathrm{id}_X - A\| = |t| \|\mathrm{id}_X - D_G P_S(x)\| \leq |t|(1+K) < 1,$$

which entails that $A$ is invertible, since all operators in $B_{\mathcal{L}(X)}(\mathrm{id}_X, 1)$ are so. We will show that $D_G P_S(y) = D_G P_S(x) \circ A^{-1}$. Fix $h \in X$, $s \in ]0,+\infty[$ and denote $u = A^{-1}h$. If $s$ is small enough such that $y + sAu \in \mathcal{W}_S(P_S(x), \gamma r, \alpha)$, we can write

$$\begin{aligned} y + sAu &= x + t(x - P_S(x)) + su + st(u - D_G P_S(x)u) \\ &= (1+t)(x + su) - t(P_S(x) + sD_G P_S(x)u). \end{aligned}$$

Also, we know that $P_S(x + su) = P_S(x) + sD_G P_S(x)u + o(s)$, and so, combining both equalities, we get that

$$y + sAu = (1+t)(x+su) - tP_S(x+su) + o(s).$$

Taking $s$ small enough, we may assume that $x + su \in \mathcal{W}_S(P_S(x), \gamma r, \alpha)$ and, by equation (5.16), that $(x+su) + t(x + su - P_S(x+su))$ is also included in $\mathcal{W}_S(P_S(x), \gamma r, \alpha)$. Thus,

$$P_S((x+su) + t(x+su - P_S(x+su))) = P_S(x+su).$$

Since $P_S(\cdot)$ is Lipschitz continuous in $\mathcal{W}_S(P_S(x), \gamma r, \alpha)$ we can write

$$P_S(y + sh) = P_S(y + sAu) = P_S(x + su) + o(s) = P_S(y) + sD_G P_S(x) \circ A^{-1}(h) + o(s),$$

where the last equality follows recalling that $P_S(y) = P_S(x)$ and $u = A^{-1}h$. Since $h$ is arbitrary, we conclude that $D_G P_S(y)$ exists and it coincides with $D_G P_S(x) \circ A^{-1}$, as we claimed. The case when $P_S$ is F-differentiable follows exactly the same proof using that $o(A^{-1}h) = o(h)$.

The second part of the proof follows directly from the bijectivity of $A$, once we note that $A(H[x]) \subseteq H[x]$. Indeed, using Lemma 5.2.2, for any $h \in H[x]$ we have that

$$\begin{aligned}\langle Ah, x - P_S(x)\rangle &= (1+t)\langle h, x - P_S(x)\rangle - t\langle D_G P_S(x)h, x - P_S(x)\rangle \\ &= (1+t)\langle h, x - P_S(x)\rangle - t\langle \Pi_{H[x]} \circ D_G P_S(x)h, x - P_S(x)\rangle \\ &= 0,\end{aligned}$$

and so, the latter inclusion holds.

Finally, suppose that $P_S(\cdot)$ is of class $\mathcal{C}^p$ near $x$. It is direct that the function $F(u) := (1+t)u - tP_S(u)$ is also of class $\mathcal{C}^p$ near $x$ and $D_F F(x) = A$. Using the Local Inverse Function Theorem, there exists a neighborhood $U \subset \mathcal{W}_S(P_S(x), \gamma r, \alpha)$ of $x$ and a neighborhood $V$ of $y = F(x)$ also included in $\mathcal{W}_S(P_S(x), \gamma r, \alpha)$ such that $F : U \to V$ is invertible and $F^{-1}$ is of class $\mathcal{C}^p$ on $V$. Since (by Proposition 4.2.2) $P_S(F(u)) = P_S(u)$ for all $u \in U$, we get that

$$P_S(v) = P_S \circ F \circ F^{-1}(v) = P_S \circ F^{-1}(v),$$

and so the conclusion follows by chain rule. The proof is complete. □

**Corollary 5.2.5** *Let $S$ be a closed body of $X$ and $x_0 \in \operatorname{bd} S$. Assume that $S$ is $r$-prox-regular at $x_0$. If $\operatorname{bd} S$ is a $\mathcal{C}^{p+1}$-submanifold at $x_0$, then there exists $\alpha > 0$ such that $P_S(\cdot)$ is of class $\mathcal{C}^p$ on $\mathcal{W}_S(x_0, r, \alpha) \setminus S$ and for each $u \in \mathcal{W}_S(x_0, r, \alpha) \setminus S$, $DP_S(u)$ restricted to $H[u]$ is invertible as a mapping from $H[u]$ onto $H[u]$.*

*Proof.* Fix $\alpha > 0$ such that $S$ is $(r, \alpha)$-prox-regular at $x_0$ and that $B_X(x_0, \alpha) \cap \operatorname{bd} S$ is a $\mathcal{C}^{p+1}$-submanifold. Choose $u \in \mathcal{W}_S(x_0, r, \alpha) \setminus S$ with $d_S(u) < \frac{r}{2}$. By Theorem 5.1.6, we know that $P_S$ is of class $\mathcal{C}^p$ near $u$. Also, noting that $H[u] = T_{P_S(u)}(\operatorname{bd} S)$, we get by Proposition 5.1.5 that $DP_S(u)$ restricted to $H[u]$ is invertible as a mapping from $H[u]$ onto $H[u]$. The rest of the proof can follow from Proposition 5.2.4. □

Observe also that Proposition 5.2.4 allows us to derive Theorem 5.1.6 from Theorem 4.4.6. We will present this parallel proof here because of its value as a different strategy.

*Alternative proof of Theorem 5.1.6.* Since $S$ is $r$-prox-regular at $x_0$, there exists $\alpha' \in ]0, \alpha]$ such that $B_X(x_0, \alpha') \cap \operatorname{int} S$ is connected, $B_X(x_0, \alpha') \cap S = B_X(x_0, \alpha') \cap \overline{\operatorname{int} S}$ and that $S$ is $(r, \alpha')$-prox-regular at $x_0$. We will show that $P_S(\cdot)$ is of class $\mathcal{C}^p$ on $\mathcal{W}_S(x_0, r, \alpha') \setminus S$, which is enough to derive the result (noting that $\mathrm{Ray}_{x_0,r}(S) \subset \mathcal{W}_S(x_0, r, \alpha') \setminus S$ for the first part and considering $\alpha' = \alpha$ for the second one).

Since $N^P(S; x) = 0$ for each $x \in B_X(x_0, \alpha') \cap \operatorname{int} S$, it is not hard to see that

$$P_S\left(\mathcal{W}_S(x_0, r, \alpha') \setminus S\right) = B_X(x_0, \alpha') \cap \operatorname{bd} S.$$

In particular, we get that $d_S$ and $d_{\operatorname{bd} S}$ (and therefore, $P_S$ and $P_{\operatorname{bd} S}$) coincide on $\mathcal{W}_S(x_0, r, \alpha') \setminus S$. Fix then $x \in B_X(x_0, \alpha') \cap \operatorname{bd} S$. Since $S$ is a closed body near $x$ (take $U = B_X(x_0, \alpha')$ in the definition of closed body) we know that

$$N^P(S; x) = \{-t\hat{n}(x) \ : \ t > 0\},$$

where $\hat{x}$ denotes the interior normal vector of $S$ at $x$, and so $\mathrm{Ray}_{x,r}(S)$ is well-defined. Since $\operatorname{bd} S$ is a $\mathcal{C}^{p+1}$-submanifold at $x$, by Theorem 4.4.6, there exists an open neighborhood $U \in \mathcal{N}_X(x)$ contained in $B_X(x_0, \alpha')$ on which $d^2_{\operatorname{bd} S}(\cdot)$ is of class $\mathcal{C}^{p+1}$. In particular, there exists $u \in U \cap \mathrm{Ray}_{x,r}(S)$ and a neighborhood $U' \in \mathcal{N}_X(u)$ contained in $U \setminus S$ on which $d^2_{\operatorname{bd} S}(\cdot)$ is of class $\mathcal{C}^{p+1}$. Thus, $P_{\operatorname{bd} S}$ is of class $\mathcal{C}^p$ on $U'$ and, since $P_S \equiv P_{\operatorname{bd} S}$ on $U'$, we can apply Proposition 5.2.4 to conclude that $P_S(\cdot)$ is of class $\mathcal{C}^p$ near each $u' \in \mathrm{Ray}_{x,r}(S)$. Finally, since

$$\mathcal{W}_S(x_0, r, \alpha') \setminus S = \mathcal{R}_S(x_0, r, \alpha') \setminus S = \bigcup_{x \in B_X(x_0,\alpha') \cap \operatorname{bd} S} \mathrm{Ray}_{x,r}(S),$$

the conclusion follows. $\square$

**Lemma 5.2.6** *Let $S$ be a closed set and let $x \in X \setminus S$. Assume that $P_S(x)$ exists, that $S$ is $(r, \alpha)$-prox-regular at $P_S(x)$ for some $r \in ]0, +\infty]$ and some $\alpha > 0$, and that $x \in \mathcal{W}_S(P_S(x), r, \alpha) \setminus S$. If $D_G P_S(x)$ exists and is surjective onto $H[x]$, then*

*(a) $D_G P_S(x)$ is invertible as a mapping from $H[x]$ onto $H[x]$.*

*(b) The partial G-derivative of $P_S$ at $P_S(x)$ with respect to $H[x]$ exists and it coincides with $\mathrm{id}_{H[x]}$.*

*Proof.* Let us denote $A := D_G P_S(x)\big|_{H[x]}$. Note by Proposition 5.2.2 that $D_G P_S(x)$ is surjective onto $H[x]$ if and only if $A$ is surjective onto $H[x]$. Fix $h \in \mathrm{Ker}(A)$. For every $z \in H[x]$ choose $y \in H[x]$ with $Ay = z$. By symmetry of $A(x)$ (see Lemma 5.2.1), we can write

$$\langle z, h\rangle = \langle Ay, h\rangle = \langle y, Ah\rangle = 0.$$

We conclude that $h = 0$, which entails the injectivity of $A$, proving the part $(a)$.

For $(b)$, fix $h \in H[x]$ and denote $u = A^{-1}h$. By hypothesis, we know that

$$P_S(x + tu) = P_S(x) + tD_G P_S(x)u + o(t).$$

Since $P_S$ is locally Lipschitz on $\mathcal{W}_S(P_S(x), r, \alpha)$, for $t > 0$ small enough we can write

$$P_S(x + tu) = P_S(P_S(x + tu)) = P_S(P_S(x) + tD_G P_S(x)u) + o(t).$$

Mixing both equalities and recalling that $D_G P_S(x)u = h$, we get that for $t > 0$ small enough

$$P_S(P_S(x) + th) = P_S(x) + th + o(t),$$

which entails, by arbitrariness of $h$, that the partial G-derivative of $P_S$ at $P_S(x)$ with respect to $H[x]$ exists and it coincides with $\mathrm{id}_{H[x]}$, finishing the proof. □

### 5.2.2 Characterization of closed bodies with smooth boundary

This section is devoted to state and prove the converse of Theorem 5.1.6, providing a full characterization of closed bodies with smooth boundary in terms of the smoothness of their metric projections. Based on the previous section, we already know that prox-regularity and epi-Lipschitz property are both necessary conditions for the smoothness of the boundary of a set. Therefore, we will work assuming that our closed set $S$ is in fact a closed body which is both prox-regular and epi-Lipschitz at some point of its boundary $\mathrm{bd}\, S$.

We will start with the case when $S$ is an epigraph itself. Consider then an open set $O \subseteq X$, a Lipschitz function $f : O \to \mathbb{R}$ with Lipschitz constant $\gamma > 0$ and a point $x_0 \in O$. Set $S := \overline{\mathrm{epi}\, f} \subset X \times \mathbb{R}$ and assume that $S$ is $(r, \alpha)$-prox-regular at $(x_0, f(x_0))$ with $\alpha > 0$ small enough such that

$$B_{X\times\mathbb{R}}((x_0, f(x_0)), \alpha) \cap S = B_{X\times\mathbb{R}}((x_0, f(x_0)), \alpha) \cap \mathrm{epi}\, f.$$

In such a case, $P_S(\cdot)$ is well-defined on $\mathcal{W}_S((x_0, f(x_0)), r, \alpha)$ and it coincides with $P_{\mathrm{epi}\, f}$ therein. Let us fix a point $(x, \lambda) \in \mathcal{W}_S((x_0, f(x_0)), r, \alpha) \setminus S$ and denote

$$\begin{aligned} \Lambda_1 &:= \pi_X \circ P_S \\ \Lambda_2 &:= \pi_{\mathbb{R}} \circ P_S. \end{aligned}$$

We will study the relationship between the smoothness of $P_S$ at $(x, \lambda)$ and the smoothness of $f$ at $\Lambda_1(x, \lambda)$. Recall that the Lipschitz property of $f$ entails equality (4.11), which guarantees that for all $(x, \lambda) \in \mathcal{W}_S((x_0, f(x_0)), r, \alpha) \setminus S$, we have that

$$\lambda - \Lambda_2(x, \lambda) = \pi_{\mathbb{R}}((x, \lambda) - P_S(x, \lambda)) < 0. \tag{5.17}$$

**Lemma 5.2.7** *Assume that the partial G-derivative of $P_S$ with respect to $H[x, \lambda]$ exists at $P_S(x, \lambda)$ and it coincides with $\mathrm{id}_{H[x,\lambda]}$. Then $f$ is G-differentiable at $\Lambda_1(x, \lambda)$.*

*Proof.* Let us denote $x_1 := \Lambda_1(x, \lambda)$ and $\hat{n} := d_S(x, \lambda)^{-1}(P_S(x, \lambda) - (x, \lambda))$. By hypothesis, on one hand we have that $P_S((x_1, f(x_1)) + th) = (x_1, f(x_1)) + th + o(t)$, for all $h \in H[x, \lambda]$.

On the other hand, since $S$ is prox-regular at $(x_1, f(x_1))$, we have that for every $(y, s) \in X \times \mathbb{R}$,

$$P_S((x_1, f(x_1)) + t(y, s)) = (x_1, f(x_1)) + tP_{T^B(S;(x_1,f(x_1)))}(y, s) + o(t).$$

By mixing both expansions of $P_S((x_1, f(x_1))+th)$, we conclude that $P_{T^B(S;(x_1,f(x_1)))}(h) = h$ for every $h \in H[x, \lambda]$, which entails that $H[x, \lambda] \subseteq T^B(S; (x_1, f(x_1)))$. Further, since $S$ is both epi-Lipschitz and tangentially regular at $(x_1, f(x_1))$, we get that $T^B(S; (x_1, f(x_1)))$ and $\operatorname{epi} f'(x; \cdot)$ both coincide with $T^C(S; (x_1, f(x_1)))$. Then, by Corollary 4.3.8, it is easy to deduce that this las cone is a half-space. This entails, considering the decomposition $X \times \mathbb{R} = H[x, \lambda] \oplus \mathbb{R}\hat{n}$

$$\operatorname{epi} f'(x_1; \cdot) = T^B(S; (x_1, f(x_1))) = \{h + s\hat{n} \ : \ h \in H[x, \lambda],\ s \geq 0\}.$$

Thus, the graph of $f'(x_1; \cdot)$ is the closed hyperplane $H[x, \lambda]$, which yields that $f'(x_1; \cdot)$ is linear and continuous. We conclude that $f$ is G-differentiable at $\Lambda_1(x, \lambda)$, finishing the proof. $\square$

**Proposition 5.2.8** *If $f$ is G-differentiable at $\Lambda_1(x, \lambda)$, then*

*(a)* $\nabla f(\Lambda_1(x, \lambda)) = -\frac{x-\Lambda_1(x,\lambda)}{\lambda-\Lambda_2(x,\lambda)}$.

*(b) The partial G-derivative of $P_S(\cdot)$ at $P_S(x, \lambda)$ with respect to*

$$T[x, \lambda] := \{(h, s) \in X \times \mathbb{R} \ : \ \langle (h, s), (\nabla f(\Lambda_1(x, \lambda)), -1)\rangle = 0\}$$

*exists and it coincides with* $\mathrm{id}_{T[x,\lambda]}$.

*Proof.* For part $(a)$, since $f$ is G-differentiable at $\Lambda_1(x, \lambda)$ we have, by Corollary 4.2.10, that $\partial_P f(A_1(x, \lambda)) \subseteq \{\nabla f(A_1(x, \lambda))\}$. Also, by equation 5.17, we know that $\lambda - \Lambda_2(x, \lambda) < 0$ and so we can write

$$\left(-\frac{x - \Lambda_1(x, \lambda)}{\lambda - \Lambda_2(x, \lambda)}, -1\right) = \frac{(x, \lambda) - P_S(x, \lambda)}{|\pi_{\mathbb{R}}((x, \lambda) - P_S(x, \lambda))|} \in N^P(S; P_S(x, \lambda)).$$

Since by construction $P_S(x, \lambda) = (\Lambda_1(x, \lambda), f(\Lambda_1(x, \lambda)))$, we deduce that

$$-\frac{x - \Lambda_1(x, \lambda)}{\lambda - \Lambda_2(x, \lambda)} \in \partial_P f(\Lambda_1(x, \lambda)),$$

which proves the desired equality.

For part $(b)$, since $f$ is tangentially regular at $\Lambda_1(x, \lambda)$ (by the prox-regularity of $S$), we can write that

$$T^B(S; P_S(x, \lambda)) = T^C(S; P_S(x, \lambda)) = T^C(\operatorname{epi} f; P_S(x, \lambda)) = \operatorname{epi} f'(\Lambda_1(x, \lambda); \cdot).$$

Since $f$ is G-differentiable at $\Lambda_1(x,\lambda)$, it results that

$$T^B(S;P_S(x,\lambda)) = \{(h,s)\in X\times\mathbb{R} \;:\; D_Gf(\Lambda_1(x,\lambda))h\le s\},$$

and hence $T[x,\lambda]\subseteq T^B(S;P_S(x,\lambda))$. Further, by Theorem 4.3.12, we know that $P_S(\cdot)$ is directionally differentiable at $P_S(x,\lambda)$ and that for every $(h,s)\in X\times\mathbb{R}$,

$$P_S(P_S(x,\lambda)+t(h,s)) = P_S(x,\lambda)+tP_{T^B(S;P_S(x,\lambda))}(h,s)+o(t).$$

This and the above inclusion $T[x,\lambda]\subseteq T^B(S;P_S(x,\lambda))$, entail that for all $(h,s)\in T[x,\lambda]$,

$$P_S(P_S(x,\lambda)+t(h,s)) = P_S(x,\lambda)+t(h,s)+o(t),$$

and so, the partial G-derivative of $P_S$ with respect to $T[x,\lambda]$ exists and it coincides with the identity $\mathrm{id}_{T[x,\lambda]}$. The proof is now complete. □

**Theorem 5.2.9** *Assume that $f$ is G-differentiable near $x_0$ and that $P_S(\cdot)$ is of class $\mathcal{C}^p$ (with $p\ge 1$) on an open neighborhood $U$ of $(\bar{x},\bar{\lambda})\in \mathrm{Ray}_{(x_0,f(x_0)),r}(S)$. Assume also that $\nabla f(x_0)=0$ and that $DP_S(u)$ is invertible in $H[u]$ for each $u\in U$. Then, $f$ is of class $\mathcal{C}^{p+1}$ near $x_0$.*

*Proof.* Let us denote $\bar{v}=(x_0,f(x_0))$. For simplicity, we will write $u=(u_1,u_2)$ for each $u\in X\times\mathbb{R}$. Without loss of generality, we may assume that $f(x_0)=0$. Denote $\bar{u}:=(\bar{x},\bar{\lambda})$ and $\hat{n}=d_S(\bar{u})^{-1}(\bar{u}-P_S(\bar{u}))$, and consider the mapping

$$\begin{aligned} F: U &\to X\times\mathbb{R}\\ u &\mapsto P_S(u)+\langle \hat{n},u-\bar{u}\rangle\,\hat{n}. \end{aligned}$$

Clearly, $F$ is of class $\mathcal{C}^p$ on $U$. Also, for all $h\in X\times\mathbb{R}$

$$DF(\bar{u})h = DP_S(\bar{u})h+\langle\hat{n},h\rangle\hat{n} = DP_S(\bar{u})\circ\Pi_{H[\bar{u}]}h+\langle\hat{n},h\rangle\hat{n},$$

and so $DF(\bar{u})$ is invertible, since $DP_S(\bar{u})$ is invertible restricted to $H[\bar{u}]$ and $\hat{n}\perp H[\bar{u}]$ by definition of $H[\bar{u}]$. Applying the Local Inverse Function Theorem, we get that there exist an open neighborhood $U_0$ of $\bar{u}$ and an open neighborhood $V$ of $F(\bar{u})=P_S(\bar{u})=\bar{v}$ such that $F:U_0\to V$ is invertible and $F^{-1}$ is of class $\mathcal{C}^p$.

Since $\nabla f(x_0)=0$ we have, by Proposition 5.2.8(a), that in fact $\hat{n}=(0,-1)$ and $H[\bar{u}]=X\times\{0\}$. Choose then an open neighborhood $V_X\in\mathcal{N}_X(x_0)$ such that $V_X\times\{0\}\subset V$ (which can be done since $f(x_0)=0$ and $V$ is a neighborhood of $(x_0,f(x_0))$) and such that $f$ is G-differentiable on $V_X$. Shrinking $U_0$ and $V$ if necessary, we may suppose that $U_0\subseteq\mathcal{W}(P_S(x_0),r,\alpha)\setminus S$, and so, provided equation (5.17), $u_2-\pi_{\mathbb{R}}(P_S(u))<0$ for each $u\in U_0$. Defining the mapping $G:V_X\to U_0$ given by $G(x):=F^{-1}(x,0)$ and using Proposition 5.2.8(a), for every $x\in V_X$ we can write

$$\begin{aligned}\nabla f(x) = \nabla f\left(\pi_X(x,0)\right) &= (\nabla f)\circ\pi_X\circ F\left(G(x)\right)\\ &= (\nabla f)\circ\Lambda_1(G(x)) = -\frac{\pi_X\circ G(x)-\Lambda_1\circ G(x)}{\pi_2\circ G(x)-\Lambda_2\circ G(x)},\end{aligned}$$

where the third equality comes from the fact that

$$\pi_X \circ F(u) = \pi_X \circ P_S(u) + \langle \hat{n}, u - \bar{u} \rangle \pi_X(\hat{n}) = \pi_X \circ P_S(u).$$

Since $G$, $\Lambda_1$ and $\Lambda_2$ are of class $\mathcal{C}^p$, we conclude that $\nabla f$ is of class $\mathcal{C}^p$ on $V_X$. This yields that $f$ is of class $\mathcal{C}^{p+1}$ on $V_X$, finishing the proof. $\square$

Now, after the development for the epigraph of Lipschitz functions, we can go back to our case of study, that is, to consider that $S$ is a closed subset of $X$.

**Proposition 5.2.10** *Let $S$ be a closed set of $X$ and let $x \in X \setminus S$. Assume that $S$ is $(r,\alpha)$-prox-regular at $P_S(x)$ for some $\alpha > 0$ and some $r \in ]0,+\infty]$ with $d_S(x) < r$. Assume also that $P_S$ is G-differentiable at $x$ and that $D_G P_S(x)\big|_{H[x]}$ is injective over $H[x]$. Then*

$$\mathbb{R}_+\{x - P_S(x)\} \subseteq N^P(S; P_S(x)) \subseteq \mathbb{R}\{x - P_S(x)\}.$$

*If in addition $S$ is epi-Lipschitz at $P_S(x)$, then*

$$\mathbb{R}_+\{x - P_S(x)\} = N^P(S; P_S(x)).$$

*Proof.* Since $P_S(x)$ exists, the inclusion $\mathbb{R}_+\{x - P_S(x)\} \subseteq N^P(S; P_S(x))$ is direct from the definition of proximal normal cone. Suppose that the second inclusion fails, that is, there exists $\xi \in N^P(S; P_S(x)) \setminus \mathbb{R}\{x - P_S(x)\}$. By definition of proximal normal cone, that means that there exists $y \in \mathcal{W}_S(P_S(x), r, \alpha) \setminus S$ such that $P_S(y) = P_S(x)$ and $\xi = y - P_S(y) \notin \mathbb{R}\{x - P_S(x)\}$. The fact that $N^P(S; P_S(x))$ is convex by Corollary 4.2.6, and the equality $P_S(x) = P_S(y)$ yield that $(x + t(y-x)) - P_S(x) \in N^P(S; P_S(x))$ for all $t \in [0,1]$, and, since we can write

$$d_S(x + t(y-x)) \leq \|x + t(y-x) - P_S(x)\| \leq t d_S(y) + (1-t) d_S(x) < r,$$

we also have that $x + t(y-x) \in \mathcal{W}_S(P_S(x), r, \alpha)$. By $(r,\alpha)$-prox-regularity we get that $P_S(x + t(y-x)) = P_S(x)$ for all $t \in [0,1]$. Using Proposition 5.2.2, this entails

$$D_G P_S(x)\left(\Pi_{H[x]}(y-x)\right) = D_G P_S(x)(y-x) = \lim_{t \downarrow 0} \frac{P_S(x + t(y-x)) - P_S(x)}{t} = 0.$$

By injectivity of $D_G P_S(x)\big|_{H[x]}$, we conclude that $\Pi_{H[x]}(y-x) = 0$. But, since $y - P_S(x) = y - P_S(y) \notin \mathbb{R}\{x - P_S(x)\} = H[x]^\perp$, we know that $\Pi_{H[x]}(y - P_S(x)) \neq 0$, and so

$$0 = \Pi_{H[x]}(y-x) = \Pi_{H[x]}(y - P_S(x)) - \Pi_{H[x]}(x - P_S(x)) = \Pi_{H[x]}(y - P_S(x)) \neq 0,$$

which is a contradiction. We conclude that $N^P(S; P_S(x)) \subseteq \mathbb{R}\{x - P_S(x)\}$ which proves the first part of the proposition.

For the second part, assume in addition that $S$ is epi-Lipschitz at $P_S(x)$. Then, we know that $T^C(S; P_S(x_0))$ has nonempty interior. If $N^P(S; P_S(x_0)) \supsetneq \mathbb{R}_+\{x - P_S(x_0)\}$, then, by the first part of this proposition and the fact that $N^P(S;P_S(x_0))$ is a cone, we would get that $N^P(S;P_S(x_0)) = \mathbb{R}\{x_0 - P_S(x_0)\}$.

Since $S$ is prox-regular at $P_S(x_0)$, we know that $N^C(S;P_S(x_0))$ coincides with $N^P(S;P_S(x_0))$, and so, since $T^C(S;P_S(x_0))$ is a closed convex cone, we could write

$$T^C(S;P_S(x_0)) = \left[N^P(S;P_S(x_0))\right]^o = \{h \in X \ : \ \langle h, x - P_S(x_0)\rangle = 0\}.$$

Since the last set in the above equality has empty interior, this yields a contradiction. Thus, $N^P(S;P_S(x_0))$ coincides $\mathbb{R}_+\{x - P_S(x_0)\}$, which finishes the proof. $\square$

Now we can state and prove the main result of this section. It is a full characterization of closed bodies with smooth boundary in terms of the smoothness of the metric projection. The converse of Theorem 5.1.6 is contained in this theorem, as implication $(c)\Rightarrow(a)$. Observe that the local prox-regularity and the epi-Lipschitz property are posed as hypotheses, but they are consequences of $(a)$.

**Theorem 5.2.11** *Let $S$ be a closed body of $X$ and let $x_0 \in \operatorname{bd} S$. Assume that $S$ is $r$-prox-regular and epi-Lipschitz at $x_0$. The following assertions are equivalent:*

*(a)* $\operatorname{bd} S$ *is a $\mathcal{C}^{p+1}$-submanifold at $x_0$.*

*(b) There exists $\alpha > 0$ such that $P_S$ is of class $\mathcal{C}^p$ on $\mathcal{W}_S(x_0,r,\alpha) \setminus S$ and for every $u \in \mathcal{W}_S(x_0,r,\alpha)\setminus S$, $DP_S(u)$ restricted to $H[x]$ is invertible as a mapping from $H[u]$ onto $H[u]$.*

*(c) There exists a neighborhood $U$ of $x_0$ such that $P_S$ is of class $\mathcal{C}^p$ on $U \setminus S$ and for every $u \in U \setminus S$, $DP_S(u)$ is surjective onto $H[u]$.*

*Proof.* $(a)\Rightarrow(b)$ is contained in Theorem 5.1.6 and $(b)\Rightarrow(c)$ is obvious, setting $U = \mathcal{W}_S(x_0,r,\alpha)$. Let us then prove $(c)\Rightarrow(a)$. By the epi-Lipschitz property, we know that there exists a nonzero vector $\nu \in N^C(S;x_0)$ small enough such that $\bar{x} := x_0 + \nu \in U$. Since $S$ is prox-regular at $x_0$, we get by normal regularity that $N^P(S;x_0) = N^C(S;x_0)$ and so $\nu \in N^P(S;x_0)$. By shrinking the norm of $\nu$ if necessary, we may assume that $P_S(\bar{x}) = x_0$.

Now, since $\bar{x} \in U$, by Lemma 5.2.6 we get that $DP_S(\bar{x})\big|_{H[\bar{x}]}$ is invertible and so by Proposition 5.2.10, we get that

$$N^C(S;x_0) = N^P(S;x_0) = \mathbb{R}_+\{\bar{x} - P_S(\bar{x})\} = \mathbb{R}_+\{\nu\}. \tag{5.18}$$

In particular, it ensues that $T^C(S;x_0) = \{h \in X \ : \ \langle h,\nu\rangle \le 0\}$ and that $\hat{n} := -\|\nu\|^{-1}\nu \in I(S;x_0)$. Denote $Z = \{\hat{n}\}^\perp$ and $z_0 = \pi_Z(x_0)$. By the epi-Lipschitz property, there exist an open neighborhood $O \in \mathcal{N}_Z(z_0)$, a real $\varepsilon > 0$ and a Lipschitz function $f : O \to \mathbb{R}$ such that the set

$$U_0 = L\left(O \times \left]f(z_0) - \varepsilon, f(z_0) + \varepsilon\right[\right)$$

is contained in $U$ and $S \cap U_0 = L(\operatorname{epi} f) \cap U_0$, where $L : Z \times \mathbb{R} \to X$ is the canonic isomorphism given by $L(z,\lambda) = z + \lambda\hat{n}$. Denote $S' := \overline{\operatorname{epi} f}$ and note that $L^{-1}(x_0) = (z_0, f(z_0))$. Since $L$ is an isometry and $P_S$ is continuous on $U_0$, it is easy to verify that there exists an open neighborhood $U_1 \in \mathcal{N}_X(x_0)$ contained in $U_0$ such that, denoting $W := L^{-1}(U_1)$,

$$P_{S'}\big|_W = P_{\operatorname{epi} f}\big|_W = L^{-1} \circ P_S \circ L\big|_W.$$

In particular, $P_{\operatorname{epi} f}$ is well-defined on $W$ and it is of class $\mathcal{C}^p$ on $W \setminus S'$. Also, for each $(z,\lambda) \in W \setminus S'$, it is clear that

$$DP_{\operatorname{epi} f}(z,\lambda) = L^{-1} \circ DP_S(L(z,\lambda)) \circ L$$

and that

$$H_{S'}[z,\lambda] = \{(z',\lambda') \in X \times \mathbb{R} \; : \; \langle L^{-1}(L(z,\lambda) - P_S(L(z,\lambda))), (z',\lambda')\rangle = 0\} = L^{-1} H_S[L(z,\lambda)].$$

In particular, we get that $DP_{\operatorname{epi} f}(z,\lambda)\big|_{H_{S'}[z,\lambda]}$ is invertible as a mapping from $H_{S'}[z,\lambda]$ onto itself. Also, since by Theorem 4.3.10 $S'$ is prox-regular at $(z_0, f(z_0))$ and $P_{S'}$ coincides with $P_{\operatorname{epi} f}$ in $W$, we may suppose by shrinking $W$ if necessary and by using Lemma 5.2.6, that for every $(z,\lambda) \in W \setminus S'$ the partial G-Derivative of $P_{\operatorname{epi} f}$ with respect to $H_{S'}[z,\lambda]$ exists at $P_{\operatorname{epi} f}(z,\lambda)$ and it coincides with the identity $\operatorname{id}_{H_{S'}[z,\lambda]}$.

We can now apply Lemma 5.2.7 to deduce that for every $(z,\lambda) \in W \setminus S'$, the function $f$ is G-differentiable at $\Lambda_1(z,\lambda)$, where $\Lambda_1 := \pi_Z \circ P_{\operatorname{epi} f}$. We claim that $\Lambda_1(W \setminus S')$ is a neighborhood of $z_0$. Indeed, fix $\alpha > 0$ and $r' > 0$ small enough such that $B_{Z\times\mathbb{R}}((z_0, f(z_0)), \alpha) \subset W$ and such that $S'$ is $(r',\alpha)$-prox-regular at $(z_0, f(z_0))$.

Since $f$ is Lipschitz continuous, we can fix $\alpha' > 0$ such that $(z, f(z)) \in B_{Z\times\mathbb{R}}((z_0, f(z_0)), \alpha)$ for each $z \in B_Z(z_0, \alpha')$. Then, for any such a $z$, since $N^P(S'; (z, f(z)))$ is nontrivial (according to the normal regularity and epi-Lipschitz property of $S'$), we get that there exists $\xi \in N^P(S'; (z, f(z)))$ such that $\|(z, f(z)) + \xi - (z_0, f(z_0))\| < \alpha$ and $(z, f(z)) = P_{S'}((z, f(z)) + \xi)$. Hence, $(z, f(z)) + \xi \in B_{Z\times\mathbb{R}}((z_0, f(z_0)), \alpha) \setminus S'$ and $\Lambda_1((z, f(z)) + \xi) = z$. This entails the inclusion

$$\Lambda_1(W \setminus S') \supseteq \Lambda_1(B_{Z\times\mathbb{R}}((z_0, f(z_0)), \alpha) \setminus S') \supseteq B_Z(z_0, \alpha'),$$

proving our claim. It then follows that $f$ is G-differentiable near $z_0$. Furthermore, noting that $L^* = L^{-1}$, we can write using Proposition 4.2.14(c)

$$\begin{aligned}
N^P(\operatorname{epi} f; (z_0, f(x_0))) &= N^C(\operatorname{epi} f \cap L^{-1}(U_0); (z_0, f(x_0))) = N^C(L^{-1}(S \cap U_0), (z_0, f(z_0))) \\
&= L^*\left[N^C(S \cap U_0; x_0)\right] \\
&= L^{-1}\left[N^C(S \cap U_0; x_0)\right] \\
&= L^{-1}\left[N^P(S; x_0)\right] = \{(\pi_Z(\xi), \langle \xi, \hat{n}\rangle) \; : \; \xi \in N^P(S; x_0)\}.
\end{aligned}$$

So using the equality $N^P(S; x_0) = \mathbb{R}_+\{(0,-1)\}$ due to equation (5.18), we see that

$$N^P(\operatorname{epi} f; (z_0, f(z_0))) = \mathbb{R}_+\{(-\pi_Z(\hat{n}), -1)\} = \mathbb{R}_+\{(0,-1)\},$$

which tells us that $\nabla f(z_0) = 0$. All the hypotheses of Theorem 5.2.9 are then fulfilled, which guarantees that $f$ is of class $\mathcal{C}^{p+1}$ near $z_0$. Finally, noting that $\operatorname{bd} S \cap U_0 = L(\operatorname{gph} f) \cap U_0$, we conclude by Proposition 4.1.4 that $\operatorname{bd} S$ is a $\mathcal{C}^{p+1}$-submanifold at $x_0$. The proof is now complete. □

### 5.2.3 Partial converse of Poly-Raby's Theorem

In the line of the development in the previous subsection leading to the smoothness of metric projection to a closed body whose boundary is a submanifold, our aim in this final subsection is to study the case when the set is itself a submanifold, in view of a converse of Theorem 4.4.6.

First, let us recall that Poly and Raby proved this converse in the finite-dimensional setting. To be more precise, they proved that, if for a closed set $M$ in a finite-dimensional Euclidean space there exists a neighborhood $U$ of a point $m_0 \in M$ on which $d_M^2(\cdot)$ is of class $\mathcal{C}^{p+1}$, then $M$ is itself a $\mathcal{C}^{p+1}$-submanifold at $m_0$ (see [25, Section 3]).

Unfortunately, this part of their proof cannot be directly extended to the infinite-dimensional setting (at least to our knowledge). Nevertheless, their main ideas can be followed if we assume in addition that the $(p+1)$-derivative of $d_M^2(\cdot)$ is uniformly continuous near $m_0$. We will present here this partial converse of Theorem 4.4.6 which, by Proposition 5.2.4, entails the same partial result for Theorem 5.1.10. To do so, we will need several lemmas.

Let us introduce the notion of locally uniform continuity. Let $E$ and $F$ be two Banach spaces and let $U$ be a nonempty open set of $E$. We will denote by $\mathcal{C}^{k,0}_{\text{loc}}(U;F)$ (with $k \geq 0$) the set of $\mathcal{C}^k$-mappings $f : U \to F$ such that the $k$th-derivative is *locally uniformly continuous*, that is, for every $u_0 \in U$ there exists $\delta_0 > 0$ with $B_X(u_0, \delta_0) \subseteq U$ such that $D^k f$ is uniformly continuous on $B_X(u_0, \delta_0)$, that is,

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall u, u' \in B_X(u_0, \delta_0),\ \|u - u'\| \leq \delta \Rightarrow \|D^k f(u) - D^k f(u')\| \leq \varepsilon.$$

Analogously, we will denote by $\mathcal{C}^{k,1}_{\text{loc}}(U;F)$ the set of $\mathcal{C}^k$-mappings $f : U \to F$ such that the $k$th-derivative is *locally Lipschitz-continuous*.

**Lemma 5.2.12** *Let $X_1$, $X_2$ and $X_3$ be three Banach spaces and let $U$ and $V$ be two nonempty sets of $X_1$ and $X_2$, respectively. Consider two mappings $f : U \to V$ and $g : V \to X_3$ such that $f \in \mathcal{C}^{p,0}_{\text{loc}}(U;X_2)$ and $g \in \mathcal{C}^{p,0}_{\text{loc}}(V;X_3)$. Then, we have that*

$$g \circ f \in \mathcal{C}^{p,0}_{\text{loc}}(U;X_3).$$

*Proof.* Let us first note that the result is direct for $k = 0$. We will prove the case $k \geq 1$ by induction. By chain rule, we know that for each $u \in U$,

$$D(g \circ f)(u) = Dg(f(u)) \circ Df(u) = \beta(Dg(f(u)), Df(u)),$$

where $\beta : \mathcal{L}(X_2, X_3) \times \mathcal{L}(X_1, X_2)$ is the bounded bilinear operator given by $\beta(T_1, T_2) = T_2 \circ T_1$. Let us define $\Psi : U \to E := \mathcal{L}(X_2, X_3) \times \mathcal{L}(X_1, X_2)$ given by $\Psi(u) = (h(u), Df(u))$, with $h := (Dg) \circ f$. We claim that $\Psi$ is locally uniformly continuous. Indeed, since $Dg(\cdot)$ and $f(\cdot)$ are locally uniformly continuous, we know that $h(\cdot)$ also is. Thus, since $Df(\cdot)$ is also locally uniformly continuous, for each $u_0 \in U$, there exists $\delta_0 > 0$ such that for all $\varepsilon > 0$ we can find $\delta_1, \delta_2 > 0$ satisfying

$$\begin{aligned} &\forall u, u' \in B_{X_1}(u_0, \delta_0),\ \|u - u'\| < \delta_1 \Rightarrow \|h(u) - h(u')\| < \varepsilon/2; \\ &\forall u, u' \in B_{X_1}(u_0, \delta_0),\ \|u - u'\| < \delta_2 \Rightarrow \|Df(u) - Df(u')\| < \varepsilon/2. \end{aligned}$$

Endowing $E$ with the 1-norm, we have that for all $u, u' \in B_{X_1}(u_0, \delta_0)$

$$\|u - u'\| < \delta \Rightarrow \|\Psi(u) - \Psi(u')\| = \|h(u) - h(u')\| + \|Df(u) - Df(u')\| < \varepsilon.$$

where $\delta = \min\{\delta_1, \delta_2\}$. By arbitrariness of $\varepsilon$ and $u_0$, we conclude that $\Psi$ is locally uniformly continuous, as claimed. Since $\beta$ is locally uniformly continuous (in fact, is of class $\mathcal{C}^\infty$) we conclude that $D(g \circ f)(\cdot) = \beta \circ \Psi(\cdot)$ is also locally uniformly continuous, which proves the result for $k = 1$.

Now, inductively, assume that for $n \geq 2$ the result holds true for $k \leq n$ and consider $k = n + 1$. It is not hard to see that $\Psi \in \mathcal{C}^{k-1,0}_{\mathrm{loc}}(U; E)$. Indeed, noting that, by induction, $h \in \mathcal{C}^{k-1,0}_{\mathrm{loc}}(U; \mathcal{L}(X_2; X_3))$ and that $Df \in \mathcal{C}^{k-1,0}_{\mathrm{loc}}(U; \mathcal{L}(X_1; X_2))$ we can replicate the above development, to conclude the locally uniform continuity of $D^{k-1}\Psi(\cdot) = (D^{k-1}h(\cdot), D^k f(\cdot))$. Then, since $\beta \in \mathcal{C}^{k-1,0}_{\mathrm{loc}}(E; \mathcal{L}(X_1; X_3))$, we conclude that

$$D(g \circ f)(\cdot) = \beta \circ \Psi(\cdot) \in \mathcal{C}^{k-1,0}_{\mathrm{loc}}(U; \mathcal{L}(X_1; X_3)),$$

which entails that $g \circ f \in \mathcal{C}^{k,0}_{\mathrm{loc}}(U; X_3)$, finishing the proof. □

Now, given two Banach spaces $E$ and $F$, assume that we have a $\mathcal{C}^p$-diffeomorphism $\Psi : U \to V$ where $U \subseteq E$ and $V \subseteq F$ are open sets. It is not hard to see that for every $v \in V$ we can write

$$D\Psi^{-1}(v) = \big(D\Psi(\Psi^{-1}(v))\big)^{-1} = J \circ D\Psi \circ \Psi^{-1}(v),$$

where $J : \mathrm{Iso}(E; F) \to \mathrm{Iso}(F; E)$ is the homeomorphism given by $J(T) := T^{-1}$. It is well-known that $\mathrm{Iso}(E; F)$ and $\mathrm{Iso}(F; E)$ are open subsets of $\mathcal{L}(E; F)$ and $\mathcal{L}(F; E)$, respectively, and that $J$ is a $\mathcal{C}^\infty$-mapping, and so it belongs to $\mathcal{C}^{p,0}_{\mathrm{loc}}(\mathrm{Iso}(E; F); \mathcal{L}(F; E))$ (see, e.g., [15, Proposition 3.1.3 and Theorem 3.1.5]). Therefore, using Lemma 5.2.12, we get that if $\Psi \in \mathcal{C}^{p,0}_{\mathrm{loc}}(U; F)$, then $D\Psi^{-1} \in \mathcal{C}^{p-1,0}_{\mathrm{loc}}(V; \mathcal{L}(F; E))$ and so $\Psi^{-1} \in \mathcal{C}^{p,0}_{\mathrm{loc}}(V; E)$. This yields, by changing the roles of $\Psi$ and $\Psi^{-1}$ when it is necessary, the following corollary:

**Corollary 5.2.13** *Let $E$ and $F$ be two Banach spaces and let $U$ and $V$ be two nonempty open sets of $E$ and $F$, respectively. If a mapping $\Psi : U \to V \subset$ is a $\mathcal{C}^p$-diffeomorphism (with $p \geq 1$), then*

$$\Psi \in \mathcal{C}^{p,0}_{\mathrm{loc}}(U; F) \iff \Psi^{-1} \in \mathcal{C}^{p,0}_{\mathrm{loc}}(V; E).$$

**Remark 5.2.14** Lemma 5.2.12 and Corollary 5.2.13 remain true if we replace the locally uniform continuity of the $p$th-derivative by the local Lipschitz property.

In view of Proposition 4.1.4, we can introduce the following two stronger versions of $\mathcal{C}^p$-submanifolds, for which we will state a characterization in terms of the differentiability of the metric projection:

**Definition 5.2.15** *Let $X$ be a Hilbert space and $p \geq 1$. A subset $M$ of $X$ is a $\mathcal{C}^{p,0}$-*submanifold *(resp. $\mathcal{C}^{p,1}$-submanifold) at a point $m_0 \in M$ if there exist a closed subspace $Z$, two open neighborhoods $U \in \mathcal{N}_X(m_0)$ and $V_Z \in \mathcal{N}_Z(0)$, and a mapping $\theta : V_Z \to Z^\perp$ such that*

*(i) $\theta$ belongs to $\mathcal{C}^{p,0}_{\mathrm{loc}}(V; Z^\perp)$ (resp. belongs to $\mathcal{C}^{p,1}_{\mathrm{loc}}(V; Z^\perp)$);*

*(ii) $\theta(0) = 0$ and $D\theta(0) = 0$; and*

*(iii) $M \cap U = (L^{-1}(\operatorname{gph}\theta) + m_0) \cap U$, where $L : X \to Z \times Z^\perp$ is the linear isomorphism given by $L(x) = (\pi_Z(x), \pi_{Z^\perp}(x))$.*

The following lemma is the fundamental piece that allows us to translate the proof of Poly and Raby into the finite-dimensional setting, for $\mathcal{C}^{p+1,0}$-submanifolds and $\mathcal{C}^{p+1,1}$-submanifolds.

**Lemma 5.2.16** *Let $X$, $Y$ and $Z$ be three Hilbert spaces and let $U \in \mathcal{N}_X(0)$ be an open neighborhood of $0$ in $X$. Consider a continuous mapping $T : U \subset X \to \mathcal{L}(Y; Z)$ and an integer $p \geq 1$. Define the mapping $g : U \times Y \to Z$ given by*

$$g(u, y) := T(u)y.$$

*Then $T$ is of class $\mathcal{C}^p$, whenever $g$ is of class $\mathcal{C}^p$ and there exists $V \in \mathcal{N}_Y(0)$ such that the family $\{D^p g(\cdot, v)\}_{v\in V}$ is* locally equi-uniformly continuous*, that is, for every $u_0 \in U$, there exists $\delta_0 > 0$ with $B_X(u_0, \delta_0) \subseteq U$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ for which we have that*

$$\forall u, u' \in B_X(u_0, \delta_0),\ \|u - u'\| \leq \delta \ \Rightarrow \sup_{v\in V} \|D^p g(u, v) - D^p g(u', v)\| \leq \varepsilon. \tag{5.19}$$

*Moreover, if $g\big|_V \in \mathcal{C}^{p,0}_{\mathrm{loc}}(U \times V; Z)$ (respectively, $g\big|_V \in \mathcal{C}^{p,1}_{\mathrm{loc}}(U \times V; Z)$), then $T \in \mathcal{C}^{p,0}_{\mathrm{loc}}(U; \mathcal{L}(Y; Z))$ (resp. $T \in \mathcal{C}^{p,1}_{\mathrm{loc}}(U; \mathcal{L}(Y; Z))$).*

*Proof.* Assume that $g$ is of class $\mathcal{C}^p$ and that there exists a neighborhood $V \in \mathcal{N}_Y(0)$ such that the family $\{D^p g(\cdot, v)\}_{v\in V}$ is locally equi-uniformly continuous. Without loss of generality, we may assume that $V = B_Y[0, \eta]$, for some $\eta > 0$. Observe that for every $k \in \{1, \dots, p-1\}$, the continuous differentiability of $D^k g$ entails that the family $\{D^k g(\cdot, v)\}_{v\in B_Y[0,\eta]}$ is also locally equi-uniformly continuous.

For every $k \in \{1, \ldots, p\}$, let us define $G_k : U \to \mathcal{L}(X^k; \mathcal{L}(Y; Z))$ as the mapping given by

$$G_k(u)(x_1, \ldots, x_k)y := D^k g(u, y)((x_1, 0), \ldots, (x_k, 0)), \quad \forall (x_1, \ldots, x_k) \in X^k, \forall y \in Y.$$

Let us show first that $G_k$ is continuous for every $k \in \{1, \ldots, p\}$. To simplify notation, for $x \in X^k$ let us denote $\tilde{x} := ((x_1, 0), \ldots, (x_k, 0)) \in X^k \times Y^k$. Now, choose $\varepsilon > 0$ and $u \in U$. For $d \in X$ we have that

$$\begin{aligned}
\|G_k(u+d) - G_k(u)\| &= \sup_{x \in \mathbb{B}_{X^k}} \|G_k(u+d)(x) - G_k(u)(x)\|_{\mathcal{L}(Y;Z)} \\
&= \sup_{y \in \mathbb{B}_Y} \sup_{x \in \mathbb{B}_{X^k}} \|G_k(u+d)(x)y - G_k(u)(x)y\|_Z \\
&= \tfrac{1}{\eta} \sup_{y \in B_Y[0,\eta]} \sup_{x \in \mathbb{B}_{X^k}} \|D^k g(u+d, y)\tilde{x} - D^k g(u, y)\tilde{x}\|_Z \\
&\le \tfrac{1}{\eta} \sup_{y \in B_Y[0,\eta]} \|D^k g(u+d, y) - D^k g(u, y)\|_{\mathcal{L}(X^k;Z)}
\end{aligned}$$

Now, by equation (5.19), we can choose $\delta > 0$ small enough (depending on $u$, $\varepsilon$ and $k$) such that

$$\sup_{y \in B_Y[0,\eta]} \|D^k g(u+d, y) - D^k g(u, y)\|_{\mathcal{L}(X^k;Z)} \le \eta\varepsilon, \quad \forall d \in B_X(0, \delta). \tag{5.20}$$

This yields that for every $d \in B_X(0, \delta)$ we have that $\|G_k(u+d) - G_k(u)\| \le \varepsilon$, which, by the arbitrariness of $\varepsilon$ and $u$, proves the continuity of $G_k$.

Now, we will prove by induction on $k \in \{1, \ldots, p\}$ that $D^k T = G_k$. Start with $k = 1$. Fix again $u \in U$ and $\varepsilon > 0$. As we did before, we can choose $\delta > 0$ small enough depending on $u$, $\varepsilon$ and $k$, such that inequality (5.20) holds for $k = 1$. Thus, for every $d \in B_X(0, \delta)$ we can write

$$\begin{aligned}
\|T(u+d) - T(u) - G_1(u)(d)\| &= \frac{1}{\eta} \sup_{y \in B_Y[0,\eta]} \|T(u+d)y - T(u)y - G_1(u)(d)y\|_Z \\
&= \frac{1}{\eta} \sup_{y \in B_Y[0,\eta]} \|g(u+d, y) - g(u, y) - Dg(u, y)(d, 0)\|_Z \\
&= \frac{1}{\eta} \sup_{y \in B_Y[0,\eta]} \|Dg(\xi, y)(d, 0) - Dg(u, y)(d, 0)\|_Z \\
&\le \left( \frac{1}{\eta} \sup_{y \in B_Y[0,\eta]} \|Dg(\xi, y) - Dg(u, y)\|_{\mathcal{L}(X;Z)} \right) \|d\| \le \varepsilon \|d\|,
\end{aligned}$$

where $\xi \in [u, u+d]$ is given by the Mean Value Theorem and the last inequality is due to the inclusion $\xi - u \in B_X(0, \delta)$. Since $G_1(u) \in \mathcal{L}(X; \mathcal{L}(Y; Z))$ and $\varepsilon$ is arbitrary, we deduce that $DT(u)$ exists and it coincides with $G_1(u)$. Since $u \in U$ is arbitrary, the conclusion follows.

Now, suppose that the result holds true for $1, \dots k-1$ with $1 < k \leq p$. We then have that $\tilde{T} := D^{k-1}T$ exists and it coincides $G_{k-1}$. Then, for $\varepsilon > 0$ and $u \in U$ we can choose, as before, $\delta > 0$ small enough such that for all $d \in B_X(0,\delta)$

$$\begin{array}{rc} & \|\tilde{T}(u+d) - \tilde{T}(u) - G_k(u)\| \\ = & \frac{1}{\eta}\sup_{y\in B_Y[0,\eta]}\sup_{x\in\mathbb{B}_{X^{k-1}}}\|\tilde{T}(u+d)(x)y - \tilde{T}(u)(x) - G_k(u)(x)y\|_Z \\ = & \frac{1}{\eta}\sup_{y\in B_Y[0,\eta]}\sup_{x\in\mathbb{B}_{X^{k-1}}}\|D^k g(\xi,y)(\tilde{x},(d,0)) - D^k g(u,y)(\tilde{x},(d,0))\|_Z \\ = & \left(\frac{1}{\eta}\sup_{y\in B_Y[0,\eta]}\|D^k g(\xi,y) - D^k g(u,y)\|_{\mathcal{L}(X^k;Z)}\right)\|d\| \\ \leq & \varepsilon\|d\|, \end{array}$$

where, again, $\xi \in [u, u+d]$ is given by the Mean Value Theorem, the last inequality is due to the inclusion $\xi - u \in B_X(0,\delta)$, and $\tilde{x} := ((x_1,0),\dots,(x_{k-1},0))$ for all $x \in X^{k-1}$. We deduce as before that $D^kT$ exists and it coincides with $G_k$. We conclude that $T$ is of class $\mathcal{C}^p$, which finishes the first part of the proof.

Now, assume that $g\big|_V \in \mathcal{C}^{p,0}_{\mathrm{loc}}(U\times V;Z)$. It is not hard to realize that the local uniform continuity of $D^p g\big|_V$ entails the local equi-uniform continuity of the family $\{D^p g(\cdot,v)\}_{v\in V}$. Thus, $D^pT$ exists, is continuous and it coincides with $G_p$. Let us show then that $G_p$ is locally uniformly continuous.

Fix then $u_0 \in U$ and let $\delta_0 > 0$ small enough such that $B_{X\times Y}((u_0,0),\delta_0) \subseteq U \times V$ and such that $D^p\left(g\big|_V\right)$ is uniformly continuous on $B_{X\times Y}((u_0,0),\delta_0)$. Note that $D^p\left(g\big|_V\right)$ coincides with $D^p g$ in $B_{X\times Y}((u_0,0),\delta_0)$. Choose $\varepsilon > 0$ and let $\delta > 0$ such that for all $(u,y),(u',y') \in B_{X\times Y}((u_0,0),\delta_0)$ we have that

$$\|(u',y')-(u,y)\| \leq \delta \implies \|D^p g(u',y') - D^p g(u,y)\|_{\mathcal{L}(X^p;Z)} \leq \frac{\delta_0\varepsilon}{2}.$$

Take any $u,u' \in B_X(u_0,\delta_0/2)$. Since $\|(x,y)\| = (\|x\|^2+\|y\|^2)^{1/2}$ for all $(x,y)\in X\times Y$, we get that $B_X(u_0,\delta_0/2)\times B_Y(0,\delta_0/2) \subseteq B_{X\times Y}((u_0,0),\delta_0)$ and so, if $\|u-u'\|\leq\delta$ we can write

$$\|G_p(u) - G_p(u')\| = \frac{2}{\delta_0}\sup_{y\in B_Y(0,\delta_0/2)}\|D^p g(u,y) - D^p g(u',y)\| \leq \varepsilon.$$

Since $\varepsilon$ is arbitrary, we deduce that $G_p$ is uniformly continuous on $B_X(u_0,\delta_0/2)$. Noting that the uniform continuity of $G_p$ near $u_0$ holds for all $u_0 \in U$, we conclude that $T \in \mathcal{C}^{p,0}_{\mathrm{loc}}(U;\mathcal{L}(Y;Z))$. The proof that $D^pT$ is locally Lipschitz whenever $g\big|_V \in \mathcal{C}^{p,1}_{\mathrm{loc}}(U\times V;Z)$ is analogous. The proof is now complete. □

**Theorem 5.2.17** *Let $M$ be a closed set of $X$ and let $m_0 \in M$. The following holds:*

*(a) $M$ is a $\mathcal{C}^{p+1,0}$-submanifold near $m_0$ if and only if there exists a neighborhood $U \in \mathcal{N}(m_0)$ such that*

*(a.i) $d_M^2(\cdot)$ is of class $\mathcal{C}^{p+1,0}$ on $U$;*

*(a.ii)* $P_M$ *is well-defined on* $U$ *and it belongs to* $\mathcal{C}^{p,0}_{\mathrm{loc}}(U;X)$.

*(b)* $M$ *is a* $\mathcal{C}^{p+1,1}$*-submanifold near* $m_0$ *if and only if there exists a neighborhood* $U \in \mathcal{N}(m_0)$ *such that*

*(b.i)* $d^2_M(\cdot)$ *is of class* $\mathcal{C}^{p+1,1}$ *on* $U$;

*(b.ii)* $P_M$ *is well-defined on* $U$ *and it belongs to* $\mathcal{C}^{p,1}_{\mathrm{loc}}(U;X)$.

*Proof.* Without loss of generality, we may assume that $m_0 = 0$. In the proof of Theorem 4.4.6, we have that whenever $M$ is a $\mathcal{C}^{p+1}$-submanifold at 0, then there exists a neighborhood $U \in \mathcal{N}_X(0)$ on which we can write

$$P_M(u) = L^{-1}(\pi_Z \circ \varphi^{-1}(u), \theta \circ \pi_Z \circ \varphi^{-1}(u)).$$

Above $Z$, $V_Z$ and $\theta : V_Z \subseteq Z \to Z^\perp$ are the closed subspace, the neighborhood in $\mathcal{N}_Z(0)$ and the $\mathcal{C}^{p+1}$-mapping given in the definition of submanifold, and the $C^p$-mapping $\varphi : V_Z \times Z^\perp \to X$ is given by

$$\varphi(v, z_2) := L^{-1}(v - D\theta(v)^* z_2, \theta(v) + z_2),$$

where $L : X \to Z \times Z^\perp$ is the canonic isomorphism given by $L(x) = (\pi_Z(x), \pi_{Z^\perp}(x))$, and where the inverse of $\varphi$ is given by the Local Inverse Function Theorem. Noting that if $\theta \in \mathcal{C}^{p+1,0}_{\mathrm{loc}}(V_Z;Z^\perp)$ (respectively, $\theta \in \mathcal{C}^{p+1,1}_{\mathrm{loc}}(V_Z;Z^\perp)$), then $\phi \in \mathcal{C}^{p+1,0}_{\mathrm{loc}}(\phi^{-1}(U);U)$ (resp. $\phi \in \mathcal{C}^{p+1,1}_{\mathrm{loc}}(\phi^{-1}(U);U)$), we get by Lemma 5.2.12 and Corollary 5.2.13 that $P_M \in \mathcal{C}^{p+1,0}_{\mathrm{loc}}(U;X)$ (resp., using Remark 5.2.14, that $P_M \in \mathcal{C}^{p+1,1}_{\mathrm{loc}}(U;X)$). Thus, the necessity in statements $(a)$ and $(b)$ follows.

Now, let us prove the sufficiency. Since the proofs of $(a)$ and $(b)$ are analogous, we will only show part $(a)$. Fix $\delta > 0$ small enough such that $d^2_M(\cdot) \in \mathcal{C}^{p+1,0}_{\mathrm{loc}}(B_X(0,2\delta);\mathbb{R})$. Since $\frac{1}{2}\nabla(d^2_M)(u) = u - P_M(u)$, we get that $P_M$ is well-defined on $B_X(0,2\delta)$ and it belongs to $\mathcal{C}^{p,0}_{\mathrm{loc}}(B_X(0,2\delta);X)$. Now, for $x \in B_X(0,\delta)$, it is clear that $P_M(x) \in B_X(0,2\delta)$ and it verifies

$$P_M(P_M(x)) = P_M(x) \text{ and } P_M(0) = 0.$$

We get then that $DP_M(0) \circ DP_M(0) = DP_M(0)$ and so, since $DP_M(0)$ is also symmetric by Lemma 5.2.1, there exists a closed subspace $Z$ of $X$ such that $DP_M(0) = \Pi_Z$ (see the Introduction of Part II).

Define the $\mathcal{C}^p$-mapping $\Psi : B_X(0,\delta) \to Z \times Z^\perp$ given by

$$\Psi(x) = \big(\pi_Z(P_M(x)), \pi_{Z^\perp}(x - P_M(x))\big).$$

Note that $\Psi(0) = 0$ and $D\Psi(0) = L$, where $L : X \to Z \times Z^\perp$ is the canonic isomorphism given by $L(h) = (\pi_Z(h), \pi_{Z^\perp}(h))$. Note also that, by Lemma 5.2.12, $\Psi \in \mathcal{C}^{p,0}_{\mathrm{loc}}(B_X(0,\delta); Z \times Z^\perp)$. Since $D\Psi(0)$ is invertible, by the Local Inverse Function Theorem and by Corollary 5.2.13, there exist two open neighborhoods $U \in \mathcal{N}_X(0)$ with $U \subseteq B_X(0,\delta)$ and $V =$

$V_Z \times V_{Z^\perp} \in \mathcal{N}_{Z\times Z^\perp}(0,0)$, such that $\Psi\big|_U : U \to V$ is a $\mathcal{C}^p$-diffeomorphism and $\left(\Psi\big|_U\right)^{-1} \in \mathcal{C}^{p,0}_{\mathrm{loc}}(V,U)$. For simplicity, denote $\Psi\big|_U$ by $\Psi$. Define the $\mathcal{C}^p$-mapping $\theta : V_Z \to Z^\perp$ by

$$\theta(v_1) := \pi_{Z^\perp} \circ \Psi^{-1}(v_1, 0), \quad \forall v_1 \in V_Z.$$

Observe that $\theta(0) = 0$ and $D\theta(0) = 0$. We claim that $M \cap U = L^{-1}(\operatorname{gph}\theta) \cap U$. Indeed, for every $u \in U$, the obvious equality $\Psi(P_M(u)) = (\pi_Z \circ \Psi(u), 0)$ tells us that $\Psi(M\cap U) = V_Z \cap \{0\} = (Z \times \{0\}) \cap \Psi(U)$. Therefore, we can write

$$\begin{aligned} & u \in M \cap U \\ \iff \quad & v := (v_1, v_2) = \Psi(u) \in V_Z \times \{0\} \text{ and } \Psi^{-1}(v) = P_M(\Psi^{-1}(v)) \\ \iff \quad & (\pi_Z(u), \pi_{Z^\perp}(u)) = (\pi_Z \circ \Psi^{-1}(v_1,0), \pi_{Z^\perp} \circ \Psi^{-1}(v_1,0)) = (v_1, \theta(v_1)) \in \operatorname{gph}(\theta) \cap L(U), \end{aligned}$$

where the last inequality follows from the fact that

$$\pi_Z(\Psi^{-1}(v_1,0)) = \pi_Z \circ P_M(\Psi^{-1}(v_1,0)) = \pi_Z \circ (\Psi(\Psi^{-1}(v_1,0))) = v_1.$$

To show that $M$ is a $\mathcal{C}^{p+1,0}$-submanifold at $0$, we only need to show that the $\mathcal{C}^p$-mapping $\theta$ is in fact belongs to $\mathcal{C}^{p+1,0}_{\mathrm{loc}}(V_Z; Z^\perp)$. To do so, fix $x \in U$ and let us denote $\Psi(x) = (z_1, z_2) \in V$. By the prox-regularity of $M$ given by the fact that $d_M^2$ is of class $\mathcal{C}^2$ on $U$, we have that $x - P_M(x) \in N^P(M; P_M(x))$. Then, recalling that $L^* = L^{-1}$, we can write

$$\begin{aligned} x - P_M(x) &\in N^P(L^{-1}(\operatorname{gph}\theta); \Psi^{-1}(z_1,0)) \\ &= L^*\big(N^P(\operatorname{gph}\theta; (z_1, \theta(z_1)))\big) = L^{-1}\big\{(-D\theta(z_1)^*\xi, \xi) \ : \ \xi \in Z^\perp\big\}, \end{aligned}$$

hence $L(x - P_M(x)) = (-D\theta(z_1)^*\xi, \xi)$ for some $\xi \in Z^\perp$. Since by definition of $L$ and $\Psi$ we have $\pi_{Z^\perp}(L(x - P_M(x))) = \pi_{Z^\perp}(x - P_M(x)) = \pi_{Z^\perp} \circ \Psi(x)$, it ensues that $\xi = z_2$, and so by the equality $x - P_M(x) = \frac{1}{2}\nabla d_M^2(x) = \frac{1}{2}\nabla d_M^2(\Psi^{-1}(z_1,z_2))$ we get that

$$D\theta(z_1)^* z_2 = -\tfrac{1}{2}\pi_Z \circ (\nabla d_M^2) \circ \Psi^{-1}(z_1, z_2). \tag{5.21}$$

This equality holds for every $(z_1, z_2) \in V$. By Lemma 5.2.12, it is direct that $-\frac{1}{2}\pi_Z \circ (\nabla d_M^2) \circ \Psi^{-1} \in \mathcal{C}^{p,0}_{\mathrm{loc}}(V, Z)$, and so the mapping $g : V_Z \times Z^\perp \to Z$ given by $g(v, z_2) := D\theta(v)^* z_2$ meets the hypothesis of Lemma 5.2.16. We conclude that

$$D\theta(\cdot)^* \in \mathcal{C}^{p,0}_{\mathrm{loc}}(V_Z; \mathcal{L}(Z^\perp; Z))$$

which entails (noting that the adjoint map $* : \mathcal{L}(Z; Z^\perp) \to \mathcal{L}(Z^\perp; Z)$ is of class $\mathcal{C}^\infty$ and invertible) that $\theta \in \mathcal{C}^{p+1,0}_{\mathrm{loc}}(V_Z; Z^\perp)$. The proof is now complete. $\square$

# Bibliography - Part II


[1] R. Abraham, J. Marsden, and T. Ratiu, *Manifolds, Tensor Analysis and Applications*. Springer Verlag, 2001.

[2] E. Asplund, "Fréchet differentiability of convex functions", *Acta Mathematica*, vol. 121, pp. 31–47, 1968.

[3] J. P. Aubin and H. Frankowska, *Set-valued analysis*. Boston: Birkhäuser, 1990.

[4] A. Avez, *Calcul Différentiel*. Paris New York: Masson, 1991.

[5] M. Bounkhel and L. Thibault, "On various notions of regularity of sets in nonsmooth analysis", *Nonlinear Analysis. Theory, Methods and Applications*, vol. 48, no. 2, pp. 223–246, 2002.

[6] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*. New York London: Springer, 2011.

[7] A. Canino, "On $p$-convex sets and geodesics", *Journal of Differential Equations*, vol. 75, pp. 118–157, 1988.

[8] F. H. Clarke, "Generalized subgradients and applications", *Transactions of the American Mathematical Society*, vol. 205, pp. 247–262, 1975.

[9] ——, *Optimization and nonsmooth analysis*. New York: Wiley, 1983, Reprinted as vol. 5 of Classics in Applied Mathematics, SIAM, Philadelphia, 1990.

[10] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*. New York: Springer, 1998.

[11] G. Colombo and L. Thibault, "Prox-regular sets and applications", in *Handbook of nonconvex analysis and applications*, D. Gao and D. Motreanu, Eds., Somerville, Mass: International Press, 2010, pp. 99–182.

[12] R. Correa, D. Salas, and L. Thibault, “Smoothness of the metric projection onto nonconvex bodies in Hilbert spaces”, *Journal of Mathematical Analysis and Applications*, 2016, (to appear).

[13] M.-O. Czarnecki and L. Thibault, “Sublevel representations of epi-Lipschitz sets and other properties”, *Mathematical Programming*, 2016, (to appear).

[14] H. Federer, “Curvature measures”, *Transactions of the American Mathematical Society*, vol. 93, pp. 418–491, 1959.

[15] M. Field, *Differential calculus and its applications*. Mineola, N.Y: Dover Publications, 2012.

[16] S. Fitzpatrick and R. R. Phelps, “Differentiability of the metric projection in Hilbert space”, *Transactions of the American Mathematical Society*, vol. 170, no. 2, pp. 483–501, 1982.

[17] R. B. Holmes, “Smoothness of certain metric projections on Hilbert space”, *Transactions of the American Mathematical Society*, vol. 184, pp. 87–100, 1973.

[18] J. B. Kruskal, “Two convex counterexamples: a discontinuous envelope function and a nondifferentiable nearest-point mapping”, *Proceedings of the American Mathematical Society*, vol. 23, pp. 697–703, 1969.

[19] M. Mazade, “Ensembles localement prox-réguliers et inéquations variationnelles”, PhD thesis, Université Montpellier II, Montpellier, France, 2011.

[20] C. D. Meyer, *Matrix analysis and applied linear algebra*. Philadelphia: Society for Industrial and Applied Mathematics, 2000.

[21] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation, I: Basic Theory; II: Applications*. Berlin New York: Springer, 2006.

[22] D. Noll, “Directional differentiability of the metric projection in Hilbert space”, *Pacific journal of Mathematics*, vol. 170, no. 2, pp. 567–592, 1995.

[23] J. P. Penot, *Calculus without derivatives*. New York: Springer, 2013.

[24] R. A. Poliquin, R. T. Rockafellar, and L. Thibault, “Local differentiability of distance functions”, *Transactions of the American Mathematical Society*, vol. 352, no. 11, pp. 5231–5249, 2000.

[25] J.-B. Poly and G. Raby, “Fonction distance et sigularités”, *Bulletin des Sciences Mathématiques (2me Série)*, vol. 108, no. 2, pp. 187–195, 1984.

[26] R. T. Rockafellar, "Clarke's tangent cones and the boundaries of closed sets in $\mathbb{R}^n$", *Nonlinear Analysis: Theory, Methods and Applications*, vol. 3, no. 1, pp. 145–154, 1979.

[27] ——, "Generalized directional derivatives and subgradients of nonconvex functions", *Canadian Journal of Mathematics*, vol. 32, no. 2, pp. 257–280, 1980.

[28] D. Salas and L. Thibault, "Characterizations of nonconvex sets with smooth boundary in terms of the metric projection in Hilbert spaces", (work in progress).

[29] A. Shapiro, "Existence and differentiability of metric projections in Hilbert spaces", *SIAM Journal on Optimization*, vol. 4, no. 1, pp. 130–141, 1994.

[30] L. Thibault, *Variational Analysis and Prox-Regular sets*, (work in progress).

[31] E. Zarantonello, "Projections on convex sets in Hilbert space and spectral theory", *Contributions to Nonlinear Analysis (Proceedings of a Symposium Conducted by the Mathematics Research Center, the University of Wisconsin-Madison)*, pp. 237–424, 1971.

**Résumé:** Ce travail est divisé en deux parties: Dans la première partie, on présente un résultat d'intégration dans les espaces localement convexes valable pour une large classe des fonctions non-convexes. Ceci nous permet de récupérer l'enveloppe convexe fermée d'une fonction à partir du sous-différentiel convexe de cette fonction. Motivé par ce résultat, on introduit la classe des espaces "Subdifferential Dense Primal Determined" (SDPD). Ces espaces jouissent des conditions permettant d'appliquer le résultat ci-dessus. On donne aussi une interprétation géométrique de ces espaces, appelée la Propriété Radon-Nikodým de Faces (FRNP). Dans la seconde partie, on étudie dans le contexte d'espaces d'Hilbert, la relation entre la lissité de la frontière d'un ensemble prox-régulier et la lissité de sa projection métrique. On montre que si un corps fermé possède une frontière $\mathcal{C}^{p+1}$-lisse (avec $p \geq 1$), alors sa projection métrique est de classe $\mathcal{C}^p$ dans le tube ouvert associé à sa fonction de prox-régularité. On établit également une version locale du même résultat reliant la lissité de la frontière autour d'un point à la prox-régularité en ce point. On étudie par ailleurs le cas où l'ensemble est lui-même une $\mathcal{C}^{p+1}$-sous-variété. Finalement, on donne des réciproques de ces résultats.

**Mots Clés:** Sous-différentiel, Intégration, Propriété Radon-Nikodým de Faces, Sous-variété, Ensemble prox-régulier, Projection métrique.

**Abstract:** This work is divided in two parts: In the first part, we present an integration result in locally convex spaces for a large class of nonconvex functions which enables us to recover the closed convex envelope of a function from its convex subdifferential. Motivated by this, we introduce the class of Subdifferential Dense Primal Determined (SDPD) spaces, which are those having the necessary condition which allows to use the above integration scheme, and we study several properties of it in the context of Banach spaces. We provide a geometric interpretation of it, called the Faces Radon-Nikodým property. In the second part, we study, in the context of Hilbert spaces, the relation between the smoothness of the boundary of a prox-regular set and the smoothness of its metric projection. We show that whenever a set is a closed body with a $\mathcal{C}^{p+1}$-smooth boundary (with $p \geq 1$), then its metric projection is of class $\mathcal{C}^p$ in the open tube associated to its prox-regular function. A local version of the same result is established as well, namely, when the smoothness of the boundary and the prox-regularity of the set are assumed only near a fixed point. We also study the case when the set is itself a $\mathcal{C}^{p+1}$-submanifold. Finally, we provide converses for these results.

**Key words:** Subdifferential, Integration, Faces Radon-Nikodým Property, Submanifold, Prox-regular set, Metric Projection.